MAT 1332, Winter 2015, Assignment 5
Due Wednesday March 25 in the dropboxes by 4:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ____________________ Student Number ____________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________

QUESTION 1. Find the solution of the following augmented matrix.

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

We use elementary row operations to simplify the matrix.

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - 1/2R_1]{R_2 + 1/2R_1} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & -3/2 & 0 & -3/2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_4 - 2/3R_2]{R_2 + R_3} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-1/3R_3}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & -3 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+3R_3} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2/3R_2} \left[\begin{array}{cccc|c} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{1/2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Denoting the columns of the matrix as x_1 through x_4 we can easily obtain the solution from the simplifed matrix. We note that there are infinitely many solutions. Letting $x_2 = t$ we obtain: $x_4 = 0$, $x_3 = 1$ and $x_1 = 1 - t$.

QUESTION 2. (a) For each of the following matrices, calculate the determinant and the inverse (if it exists).

a) $A = \begin{bmatrix} -1 & 17 \\ 4 & 2 \end{bmatrix}$

We have that $\det(A) = (-1)(2) - (17)(4) = -70 \neq 0$. Thus, A is invertible and we have that

$$\left[\begin{matrix} -1 & 17 \\ 4 & 2 \end{matrix} \,\middle|\, \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \xrightarrow{R_2+4R_1} \left[\begin{matrix} -1 & 17 \\ 0 & 70 \end{matrix} \,\middle|\, \begin{matrix} 1 & 0 \\ 4 & 1 \end{matrix}\right]$$

$$\xrightarrow{R_2(\frac{1}{70})} \left[\begin{matrix} -1 & 17 \\ 0 & 1 \end{matrix} \,\middle|\, \begin{matrix} 1 & 0 \\ \frac{2}{35} & \frac{1}{70} \end{matrix}\right]$$

$$\xrightarrow{R_1-17R_2} \left[\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \,\middle|\, \begin{matrix} \frac{1}{35} & -\frac{17}{70} \\ \frac{2}{35} & \frac{1}{70} \end{matrix}\right]$$

$$\xrightarrow{R_1(-1)} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \,\middle|\, \begin{matrix} -\frac{1}{35} & \frac{17}{70} \\ \frac{2}{35} & \frac{1}{70} \end{matrix}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} -\frac{1}{35} & \frac{17}{70} \\ \frac{2}{35} & \frac{1}{70} \end{bmatrix}$$

b) $B = \begin{bmatrix} 2 & -3 \\ -5 & 7.5 \end{bmatrix}$

We have that $\det(B) = (2)(7.5) - (-3)(-5) = 0$. Therefore, B is not invertible.

c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

The determinant of C is given by

$$\det(C) = 1(40 - 0) - 2(16 - 3) + 3(0 - 5) = -1$$

Therefore, C is invertible. To find the inverse of C we form the augmented matrix $[C|I_3]$. Then we row reduce as follows:

$$[C|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\begin{bmatrix} R_2 - 2R_1 \\ R_3 - R_1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{\begin{bmatrix} R_2 + 3R_3 \\ R_1 - 3R_3 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] = [I_3|C^{-1}]$$

Thus

$$C^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

d) $D = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

$$\det(D) = 1(20 - (-2)) - 6(10 - 1) + 4(4 - (-4)) = 0$$

Therefore, D is not invertible

QUESTION 3. For each of the following matrices, find its eigenvalues and the corresponding eigenvectors.

a) $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$

For matrix A the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}\right) \\ &= (5-\lambda)(2-\lambda) - 2(2) \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 6)(\lambda - 1) \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 6$ and $\lambda = 1$.

To find the eigenvectors of A associated with the eigenvalue $\lambda = 6$ we reduce the associated augmented matrix $[A - (6)I|0]$.

$$[A - 6I|0] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}; t \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = 6$.

To find the eigenvectors of A associated with the eigenvalue $\lambda = 1$ we reduce the associated augmented matrix $[A - (1)I|0]$.

$$[A - I|0] = \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - (1/2)R_1} \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \end{bmatrix}; s \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = 1$.

b) $A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & 2 & 0 \\ 7 & 0 & 1 \end{bmatrix}$

For matrix A the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 7 & 0 & 1-\lambda \end{bmatrix} \right) \\ &= (5-\lambda)\Big((2-\lambda)(1-\lambda) - 0\Big) - 0 + 3\Big(0 - (2-\lambda)(7)\Big) \\ &= (2-\lambda)\Big((5-\lambda)(1-\lambda) - 21\Big) \\ &= (2-\lambda)(\lambda^2 - 6\lambda - 16) \\ &= (2-\lambda)(\lambda+8)(\lambda+2) \end{aligned}$$

Therefore, the eigenvalues are $\lambda = -2$, $\lambda = 8$ and $\lambda = 2$.

To find the eigenvectors of A associated with the eigenvalue $\lambda = -2$ we reduce the associated augmented matrix $[A - (-2)I|0]$.

$$[A + 2I|0] = \left[\begin{array}{ccc|c} 7 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 7 & 0 & 3 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 7 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -3/7 \\ 0 \\ 1 \end{bmatrix}; t \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = -2$.

To find the eigenvectors of A associated with the eigenvalue $\lambda = 2$ we consider the associated augmented matrix $[A - (2)I|0]$.

$$[A - 2I|0] = \left[\begin{array}{ccc|c} 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 7 & 0 & -1 & 0 \end{array}\right].$$

Thus, it is easily seen that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; s \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = 2$.

To find the eigenvectors of A associated with the eigenvalue $\lambda = 8$ we consider the associated augmented matrix $[A - (8)I|0]$.

$$[A - 8I|0] = \left[\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & -6 & 0 & 0 \\ 7 & 0 & -7 & 0 \end{array}\right] \xrightarrow{(-1/3)R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -6 & 0 & 0 \\ 7 & 0 & -7 & 0 \end{array}\right]. \xrightarrow{R_3 - 7R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, it is easily seen that

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = q \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]; q \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = 8$.

QUESTION 4. Determine the (complex-value) eigenvalues and eigenvectors of the matrix

$$B = \left[\begin{array}{cc} 2 & 9 \\ -1/2 & -1 \end{array}\right]$$

For matrix B the characteristic polynomiial is

$$\begin{aligned} p(\lambda) &= \det(B - \lambda I) \\ &= \det\left(\left[\begin{array}{cc} 2-\lambda & 9 \\ -1/2 & -1-\lambda \end{array}\right]\right) \\ &= (2-\lambda)(-1-\lambda) - 9(-1/2) \\ &= \lambda^2 - \lambda + 5/2 \end{aligned}$$

To find the eigenvalues, we use the quadratic formula:

$$\begin{aligned} \lambda &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(5/2)}}{2(1)} \\ &= \frac{1 \pm \sqrt{-9}}{2} \\ &= \frac{1}{2} \pm \frac{3}{2}i. \end{aligned}$$

Thus, the eigenvalues of B are $\lambda = \frac{1}{2} + \frac{3}{2}i$ and $\lambda = \frac{1}{2} - \frac{3}{2}i$.

To find the eigenvectors of B associated with the eigenvalue $\lambda = \frac{1}{2} + \frac{3}{2}i$ we reduce the associated augmented matrix $[B - (1/2 - 3/2i)I|0]$.

$$[B - (1/2 - 3/2i)|0] = \left[\begin{array}{cc|c} 3/2 - 3/2i & 9 & 0 \\ -1/2 & -3/2 - 3/2i & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow} \left[\begin{array}{cc|c} -1/2 & -3/2 - 3/2i & 0 \\ 3/2 - 3/2i & 9 & 0 \end{array}\right].$$

$$\xrightarrow{(-2)R_1} \left[\begin{array}{cc|c} 1 & 3+3i & 0 \\ 3/2-3/2i & 9 & 0 \end{array}\right] \xrightarrow{R_2-(3/2-3/2i)R_1} \left[\begin{array}{cc|c} 1 & 3+3i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus

$$\left[\begin{array}{c} x \\ y \end{array}\right] = t \left[\begin{array}{c} -3-3i \\ 1 \end{array}\right]; t \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = 1/2 + 3/2i$.

To find the eigenvalues of B associated with the eigenvector $\lambda = 1/2 - 3/2i$ we use the property of complex conjugates (or solve once again). Thus,

$$\left[\begin{array}{c} x \\ y \end{array}\right] = s \left[\begin{array}{c} -3+3i \\ 1 \end{array}\right]; s \neq 0$$

are the associated eigenvectors with the eigenvalue $\lambda = 1/2 - 3/2i$.

QUESTION 5. Find the unique solution of the following system of linear differential equations

$$\frac{d}{dt}\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right], \quad \left[\begin{array}{c} x(0) \\ y(0) \end{array}\right] = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$$

- We need to find the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.
- The characteristic equation is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)(2-\lambda) - 1 \\ &= \lambda^2 - 3\lambda + 1 \end{aligned}$$

- We then have that

$$\begin{aligned} \lambda &= \frac{3 \pm \sqrt{(9) - 4(1)(1)}}{2(1)} \\ &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

- Thus, the eigenvalues are $\lambda_1 = \frac{3+\sqrt{5}}{2}$ et $\lambda_2 = \frac{3-\sqrt{5}}{2}$.
- We find the eigenvectors by solving the system $[A - \lambda I|0]$ for each eigenvalue λ.

- For $\lambda = \frac{3+\sqrt{5}}{2}$:

$$\left[A - \left(\frac{3+\sqrt{5}}{2}\right) I | 0\right] = \begin{bmatrix} \frac{-1-\sqrt{5}}{2} & 1 & | & 0 \\ 1 & \frac{1-\sqrt{5}}{2} & | & 0 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} & | & 0 \\ \frac{-1-\sqrt{5}}{2} & 1 & | & 0 \end{bmatrix}$$

$$\overset{R_2 + \left(\frac{1+\sqrt{5}}{2}\right) R_1}{\sim} \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

- Thus, the family of eigenvectors associated with λ_1 is

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2} t \\ t \end{bmatrix}; t \in \mathbb{R}, t \neq 0$$

- An eigenvector in particular is obtained by setting $t = 1$,

$$\mathbf{v}_1 = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

- For $\lambda = \frac{3-\sqrt{5}}{2}$:

$$\left[A - \left(\frac{3-\sqrt{5}}{2}\right) I | 0\right] = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} & 1 & | & 0 \\ 1 & \frac{1+\sqrt{5}}{2} & | & 0 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} & | & 0 \\ \frac{-1+\sqrt{5}}{2} & 1 & | & 0 \end{bmatrix}$$

$$\overset{R_2 + \left(\frac{1-\sqrt{5}}{2}\right) R_1}{\sim} \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

- Thus, the family of eigenvectors associated with λ_2 is

$$\begin{bmatrix} \frac{-1-\sqrt{5}}{2} t \\ t \end{bmatrix}; t \in \mathbb{R}, t \neq 0$$

- An eigenvector in particular is obtained by setting $t = 1$,

$$\mathbf{v}_2 = \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

- So, we have the solutions

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \alpha e^{\frac{3+\sqrt{5}}{2} t} \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + \beta e^{\frac{3-\sqrt{5}}{2} t} \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

- We now determine the constants α et β using the initial condition

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + \beta \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

- By solving

$$\left[\begin{array}{cc|c} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} & 1 \\ 1 & 1 & -1 \end{array}\right] \underset{\sim}{R_1(2)\to R_1} \left[\begin{array}{cc|c} -1+\sqrt{5} & -1-\sqrt{5} & 2 \\ 1 & 1 & -1 \end{array}\right]$$

$$\underset{\sim}{R_1\leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ -1+\sqrt{5} & -1-\sqrt{5} & 2 \end{array}\right]$$

$$\underset{\sim}{R_2+(1-\sqrt{5})R_1} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -2\sqrt{5} & 1+\sqrt{5} \end{array}\right]$$

$$\underset{\sim}{R_2\left(-\frac{1}{2\sqrt{5}}\right)\to R_2} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -\frac{1}{2\sqrt{5}}-\frac{1}{2} \end{array}\right]$$

$$\underset{\sim}{R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}+\frac{1}{2\sqrt{5}} \\ 0 & 1 & -\frac{1}{2\sqrt{5}}-\frac{1}{2} \end{array}\right]$$

- Thus, the particular solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \left(-\frac{1}{2}+\frac{1}{2\sqrt{5}}\right) e^{\frac{3+\sqrt{5}}{2}t} \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + \left(-\frac{1}{2}-\frac{1}{2\sqrt{5}}\right) e^{\frac{3-\sqrt{5}}{2}t} \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$