MAT 1332, Winter 2015, Assignment 3
Due Thursday February 12 by 9:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1. Determine whether the following improper integrals converge or diverge.

(a) $\displaystyle\int_0^1 \frac{\ln(x)}{x}\,dx$

[3] We first evaluate the indefinite integral $\displaystyle\int \frac{\ln(x)}{x}\,dx$. This integral requires substitution: we let $u = \ln(x) \ \Rightarrow du = \frac{1}{x}dx$. Substituting the expressions for u and du into the above integral we obtain:

$$\int u\,du = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C.$$

Now, we return to the definite integral and note that $\frac{\ln(x)}{x}$ is discontinuous at zero. Re-writing the integral and making use of the above result we obtain

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{x}\,dx &= \lim_{t\to 0^+}\int_t^1 \frac{\ln(x)}{x}\,dx \\ &= \lim_{t\to 0^+}\int_t^1 \left(\frac{(\ln(x))^2}{2}\right)\Bigg|_t^1 = \lim_{t\to 0^+}\left(0 - \frac{(\ln(t))^2}{2}\right) \\ &= -\lim_{t\to 0^+}\left(\frac{(\ln(t))^2}{2}\right) = -\infty,\ \text{divergent.}\end{aligned}$$

(b) $\displaystyle\int_1^\infty (1+x)e^x\,dx$

We first evaluate the indefinite integral $\displaystyle\int (1+x)e^x\,dx$. This integral requires integration by parts: let $u = 1 + x$ and $dv = e^x dx \Rightarrow du = dx$ and $v = e^x$. Substituting these expressions into the above integral we obtain:

$$\int (1+x)e^x\,dx = (1+x)e^x - \int e^x dx = (1+x)e^x - e^x + C.$$

Re-writing the infinite integral and making use of the above result we obtain

$$\begin{aligned}\int_1^{\infty} (1+x)e^x\,dx &= \lim_{t\to\infty} \int_1^t (1+x)e^x\,dx \\ &= \lim_{t\to\infty} \Big((1+x)e^x - e^x\Big)\Big|_1^t = \lim_{t\to\infty} \Big((1+t)e^t - e^t - (2e - e)\Big)\end{aligned}$$

To determine if the integral converges or diverges, we need to calculate

$$\lim_{t\to\infty} \Big((1+t)e^t - e^t\Big) = \lim_{t\to\infty} te^t = \infty, \text{ divergent.}$$

Question 2. Show that each function is a solution of the given initial value problem.

(a) Differential equation: $y' + y = \dfrac{2}{1+4e^{2x}}$, Initial condition: $y(-\ln 2) = \dfrac{\pi}{2}$,
Solution candidate: $y = e^{-x}\arctan(2e^{x})$
We first show the solution candidate satisfies the initial condition:

$$\begin{aligned} y(-\ln(2)) &= e^{-(-\ln(2))}\arctan\left(2e^{-\ln(2)}\right) \\ &= e^{\ln(2)}\arctan\left(2e^{\ln(2)^{-1}}\right) \\ &= 2\arctan(1) = 2\frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$

Now we show the solution candidate satisfies the given differential equation. The derivative of the solution candidate is

$$y' = -e^{-x}\arctan(2e^{x}) + e^{-x}\frac{2e^{x}}{1+4e^{2x}}.$$

To show the solution candidate satisfies the given DE, we begin with the left-hand side:

$$\begin{aligned} y' + y &= -e^{-x}\arctan(2e^{x}) + \frac{2}{1+4e^{2x}} + e^{-x}\arctan(2e^{x}) \\ &= \frac{2}{1+4e^{2x}} = \text{right-hand side of the DE.} \end{aligned}$$

(b) Differential equation: $x^2y' = xy - y^2, x > 1$, Initial condition: $y(e) = e$,
Solution candidate: $y = \dfrac{x}{\ln x}$
We first show the solution candidate satisfies the initial condition:

$$y(e) = \frac{e}{\ln(e)} = \frac{e}{1} = e.$$

To show that the solution candidate satisfies the given DE requires the derivative of the solution candidate: $y' = \frac{\ln(x)-1}{x^2}$. We now consider each side of the DE separately. First, the left-hand side

$$x^2y' = \frac{x^2}{(\ln(x))^2}(\ln(x) - 1).$$

The right-hand side simplifies to

$$\begin{aligned} xy - y^2 = x\frac{x}{\ln(x)} - \frac{x^2}{(\ln(x))^2} &= \frac{x^2}{\ln(x)} - \frac{x^2}{(\ln(x))^2} \\ &= \frac{x^2}{(\ln(x))^2}(\ln(x) - 1). \end{aligned}$$

Therefore, we see the left-hand side equals the right-hand side meaning $y = \frac{x}{\ln(x)}$ is a solution to the given DE.

QUESTION 3. Find the general solution of the following differential equations.

(a) $y^2 \dfrac{dy}{dx} = 3x^2y^3 - 6x^2$

This differential equation is separable. Factoring the right-hand side, separating variables and integrating we obtain

$$\begin{aligned} y^2 \frac{dy}{dx} &= 3x^2(y^3 - 2) \\ \int \frac{y^2}{y^3 - 2}\, dy &= \int 3x^2\, dx \\ \frac{1}{3} \ln|y^3 - 2| &= x^3 + C \\ \ln|y^3 - 2| &= 3(x^3 + C) \\ |y^3 - 2| &= e^{3(x^3 + C)} = Ae^{3x^3} \\ y^3 &= \pm Ae^{3x^3} + 2 = Be^{3x^3} + 2 \\ y &= \sqrt[3]{Be^{3x^3} + 2}. \end{aligned}$$

(b) $\dfrac{dy}{dx} = \dfrac{e^{2x-7}}{e^{x+y}}$

[3] This differential equation is separable. Using laws of exponents, we simplify the right-hand side, separate variables then integrate

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{2x-7}}{e^{x+y}} \\ \frac{dy}{dx} &= \frac{e^{2x-7}}{e^x e^y} \\ \int e^y\, dy &= \int e^{2x-7} e^{-x}\, dx \\ \int e^y\, dy &= \int e^{x-7}\, dx \\ e^y &= e^{x-7} + C \\ y &= \ln\left(e^{x-7} + C\right) \end{aligned}$$

Question 4. (a). Find the general solution of the differential equations

$$y' = y\tan(\pi t)$$

This differential equation is separable. Separating variables and integrating we obtain

$$\begin{aligned}\frac{dy}{dt} &= y\tan(\pi t)\\ \int \frac{dy}{y} &= \int \tan(\pi t)\,dt\\ \int \frac{dy}{y} &= \int \frac{\sin(\pi t)}{\cos(\pi t)}\,dt,\end{aligned}$$

The integral on the right-hand side requires substitution: let $u = \cos(\pi t), du = -\pi\sin(\pi t)$

$$\begin{aligned}\int \frac{\sin(\pi t)}{u}\frac{du}{-\pi\sin(\pi t)} &= \frac{-1}{\pi}\int \frac{1}{u}\,du\\ &= \frac{-1}{\pi}\ln|u| + C = \frac{-1}{\pi}\ln\big|\cos(\pi t)\big| + C.\end{aligned}$$

Going back to the differential equation we obtain

$$\begin{aligned}\int \frac{dy}{y} &= \int \frac{\sin(\pi t)}{\cos(\pi t)}\,dt,\\ \ln|y| &= \frac{-1}{\pi}\ln\big|\cos(\pi t)\big| + C\\ |y| &= e^{\frac{-1}{\pi}\ln\left|\cos(\pi t)\right| + C} = e^{C}e^{\frac{-1}{\pi}\ln\left|\cos(\pi t)\right|} = Ae^{\frac{-1}{\pi}\ln\left|\cos(\pi t)\right|}\\ y &= \pm Ae^{\frac{-1}{\pi}\ln\left|\cos(\pi t)\right|} = Be^{\ln\left|\cos(\pi t)\right|^{\frac{-1}{\pi}}} = B|\cos(\pi t)|^{\frac{-1}{\pi}}\end{aligned}$$

Note, we may write the general solution as

$$y = \frac{\bar{B}}{(\cos(\pi t))^{\frac{1}{\pi}}}$$

since we absorb the sign into the constant.

(b) Find the particular (unique) solution satisfying the initial condition $y(0) = 3$.

We have

$$y(0) = \frac{\bar{B}}{(\cos(0))^{\frac{1}{\pi}}} = \bar{B} = 3.$$

Therefore, the unique solution is

$$y = \frac{3}{(\cos(\pi t))^{\frac{1}{\pi}}}.$$

QUESTION 5. We estimate the mass $M(t)$ in grams of a t-oad at time t in days is described by the differential equation

$$\frac{dM}{dt} = (1+t^2)e^{-2t}, t \geq 0.$$

(a) If the mass at birth is $M(0) = 1$ g, give the solution of this initial value problem.

[3] This differential equation is autonomous so we do not need to separate variables. To integrate the function on the right-hand side requires integration by parts: we let $u = 1+t^2$ and $dv = e^{-2t} \Rightarrow du = 2tdt$ and $v = \frac{e^{-2t}}{-2}$. We obtain

$$M(t) = \int (1+t^2)e^{-2t}\, dt = -\frac{(1+t^2)e^{-2t}}{2} + \int te^{-2t}\, dt.$$

The integral $\int te^{-2t}$ requires integration by parts: we let $u = t$ and $dv = e^{-2t} \Rightarrow du = dt$ and $v = \frac{e^{-2t}}{-2}$. We obtain

$$\begin{aligned} M(t) = -\frac{(1+t^2)e^{-2t}}{2} - \frac{te^{-2t}}{2} + \frac{1}{2}\int e^{-2t}\, dt &= -\frac{(1+t^2)e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{1}{4}e^{-2t} + C \\ &= -\frac{(t+1)te^{-2t}}{2} - \frac{3}{4}e^{-2t} + C. \end{aligned}$$

Now we use the initial condition $M(0) = 1$ to solve for C :

$$1 = \frac{-1}{2} + \frac{-1}{4} + C \Rightarrow C = \frac{7}{4}.$$

Therefore,

$$M(t) = -\frac{(t+1)te^{-2t}}{2} - \frac{3}{4}e^{-2t} + \frac{7}{4}.$$

(b) What is the mass of the toad when $t = 1$ day.

[1] The mass of the toad when $t = 1$ day is

$$M(1) = -\frac{1e^{-2}}{2}(2) - \frac{3}{4}e^{-2} + \frac{7}{4} = -\frac{7}{4}e^{-2} + \frac{7}{4} \approx 1.51.$$

(c) What is the greatest rate of increase of the toad?

[1] We take the derivative of $f(t) = (1+t^2)e^{-2t}$:

$$f'(t) = 2te^{-2t} - 2(1+t^2)e^{-2t} = 2e^{-2t}\big(-t^2+t-1\big).$$

We note that e^{-2t} is positive for all t and the vertex of the above quadratic occurs at $x = \frac{1}{2}$, $y = -\frac{3}{2}$. Since the quadratic is concave down and the y coordinate of the vertex is negative, $f'(t)$ is negative for all t. Therefore, $f(t)$ is a decreasing function so that the greatest rate of increase occurs at $t = 0$:

$$f(0) = 1.$$

(d) If the rate of increase of the toad was constant and equal to the maximum rate of increase found in (c), what would be the mass of the toad be after 1 day?

[1] The rate of increase, $\frac{dM}{dt}$ in this case is equal to one which means

$$\frac{dM}{dt} = 1 \Rightarrow dM = dt \Rightarrow M(t) = t + D.$$

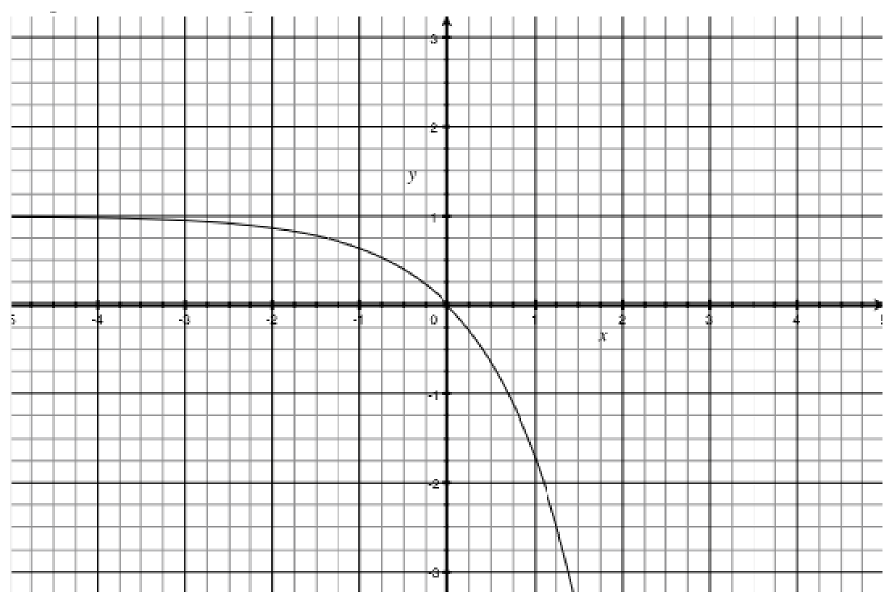
Using the initial condition, we find that $D = 1$ and, therefore, $M(t) = t+1 \Rightarrow M(1) = 2.$.

QUESTION 6. Consider the differential equation

$$\frac{dx}{dt} = 1 - e^x.$$

(a) Graph the rate of change as a function of the state variable over the interval $-2 \leq x \leq 2$.

[1]



(b) Draw the phase line diagram

[2]

(c) Give a rough sketch of some solution curves.

[2]