MAT 1332, Winter 2015, Assignment 1
Due Wednesday January 28 by 4:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ____________________ Student Number ____________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________

QUESTION 1. Calculate the following

(a) $\displaystyle\int_1^2 \frac{s^2+\sqrt{s}}{s^2}\,ds$

$$= \int_1^2 \frac{s^2}{s^2} + \frac{\sqrt{s}}{s^2}\,ds = \int_1^2 1 + s^{-3/2}\,ds = (s - 2s^{-1/2})\Big|_1^2$$
$$= 2 - 2(2^{-1/2}) - \left(1 - 2(1^{-1/2})\right) = 3 - 2^{1/2}$$

(b) $\displaystyle\int_0^\pi \frac{1}{2}\Big(\cos(x) + \big|\cos(x)\big|\Big)\,dx$

[2] We must be careful with this integral due to the absolute value. Note that $\cos(x) \geq 0$ when $x \in [0, \pi/2]$ and $\cos(x) \leq 0$ when $x \in [\pi/2, \pi]$. Therefore, the integral becomes

$$\int_0^\pi \frac{1}{2}\Big(\cos(x) + \big|\cos(x)\big|\Big)\,dx = \frac{1}{2}\int_0^\pi \cos(x)\,dx + \frac{1}{2}\int_0^\pi \big|\cos(x)\big|\,dx$$
$$= \frac{1}{2}\int_0^\pi \cos(x)\,dx + \frac{1}{2}\int_0^{\pi/2} \cos(x)\,dx - \frac{1}{2}\int_{\pi/2}^\pi \cos(x)\,dx$$
$$= \frac{1}{2}\sin(x)\Big|_0^\pi + \frac{1}{2}\sin(x)\Big|_0^{\pi/2} - \frac{1}{2}\sin(x)\Big|_{\pi/2}^\pi$$
$$= \frac{1}{2}(0+1+1) = 1.$$

QUESTION 2. Calculate the following

(a) $\displaystyle\int_0^{\pi} \frac{\cos(z)}{\sqrt{4+3\sin(z)}}\,dz$

This integral requires substitution: let

$$u = 4 + 3\sin(z) \Rightarrow du = 3\cos(z)dz \Rightarrow \frac{du}{3\cos(z)} = dz.$$

We also change the limits from z values to u values: when $z = 0, u = 4$ and when $z = \pi, u = 4$. Substituting these expressions into the integral, one obtains

$$\int_0^{\pi} \frac{\cos(z)}{\sqrt{4+3\sin(z)}}\,dz = \int_4^4 \frac{\cos(z)}{\sqrt{u}}\frac{du}{3\cos(z)}\,dz = \frac{1}{3}\int_4^4 \frac{1}{\sqrt{u}}du = 0.$$

Note: the integral equals zero since the lower and upper limits are the same.

(b) $\displaystyle\int_0^{\ln\left(\sqrt{3}\right)} \frac{e^{2x}}{1+e^{2x}}\,dx$

[3] This integral also requires substitution: let

$$u = 1 + e^{2x} \Rightarrow du = 2e^{2x}dx \Rightarrow \frac{du}{2e^{2x}} = dx.$$

We also change the limits from x values to u values: when $x = 0, u = 2$ and when $x = \ln\left(\sqrt{3}\right), u = 1 + e^{2\ln\left(\sqrt{3}\right)} = 1 + e^{\ln\left(\sqrt{3}\right)^2} = 4$. Substituting these expressions into the integral we obtain:

$$\int_0^{\ln\left(\sqrt{3}\right)} \frac{e^{2x}}{1+e^{2x}}\,dx = \int_2^4 \frac{e^{2x}}{u}\frac{du}{2e^{2x}} = \frac{1}{2}\int_2^4 \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_2^4 = \frac{1}{2}\Big(\ln(4) - \ln(2)\Big)$$

(Note that the absolutele values aren't needed in the logarithm, since it is positive over the range $2 \le u \le 4$.)

QUESTION 3. The amount of chemical produced follows $\frac{dP}{dt} = 5te^{-t}$ with initial condition $P(0) = 2$, where t is the time in minutes and P is measured in moles. How much chemical is produced between times $t = 5$ and $t = 10$?

[2] *First approach:*

The solution to this pure-time differential equation is

$$P(t) = \int 5te^{-t}\,dt = 5\int te^{-t}\,dt.$$

By using integration by parts, let $u = t$ and $dv = e^{-t}dt$, then $du = dt$ and $v = -e^{-t}$. Thus

$$P(t) = 5\int te^{-t}\,dt = 5\Big[t(-e^{-t}) - \int(-e^{-t})\,dt\Big] = 5\Big[-te^{-t} + \int e^{-t}\,dt\Big] = -5te^{-t} - 5e^{-t} + C$$

and the constant is obtained from $P(0) = -5 + C = 2$ so $C = 7$. Hence, $P(t) = -5te^{-t} - 5e^{-t} + 7$, and

$$P(5) = -25e^{-5} - 5e^{-5} + 7 \approx 6.7979, \qquad P(10) = -50e^{-10} - 5e^{-10} + 7 \approx 6.9975.$$

The amount produced between times 5 and 10 is $P(10) - P(5) \approx 0.1996$ moles.

Second Approach: Using the Fundamental Theorem of Calculus, we don't need to evaluate the constant but can compute the solution directly

$$\begin{aligned} P(10) - P(5) &= \int_5^{10} \frac{dP}{dt}\,dt \\ &= \int_5^{10} 5te^{-t}\,dt \\ &= -5te^{-t} - 5e^{-t}\Big|_5^{10} \\ &= \big(-50e^{-10} - 5e^{-10}\big) - \big(-25e^{-5} - 5e^{-5}\big) \\ &= -55e^{-10} + 30e^{-5} \approx 0.1996 \text{ moles.} \end{aligned}$$

Question 4. Calculate the total area between $y = x - x^2$, $x = -2$, $x = 2$ and the x-axis.

In order to compute the total area, we must first determine the x-intercepts of $y = x - x^2 = x(x-1)$ and these are given by $x = 0, 1$. Since $y = x - x^2$ is a negative quadratic, it is negative over the intervals $(-2, 0)$ and $(1, 2)$ and is it positive over the interval $(0, 1)$. Thus, the desired integrals to compute the total area are

$$\int_{-2}^{0} -(x - x^2)\, dx + \int_{0}^{1} (x - x^2)\, dx + \int_{1}^{2} -(x - x^2)\, dx$$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{-2}^{0} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{1} + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{1}^{2}$$

$$= 0 - \left(\frac{(-2)^3}{3} - \frac{(-2)^2}{2}\right) + \left(\frac{1}{2} - \frac{1}{3} - 0\right) + \left(\frac{2^3}{3} - \frac{2^2}{2} - \left[\frac{1}{3} - \frac{1}{2}\right]\right)$$

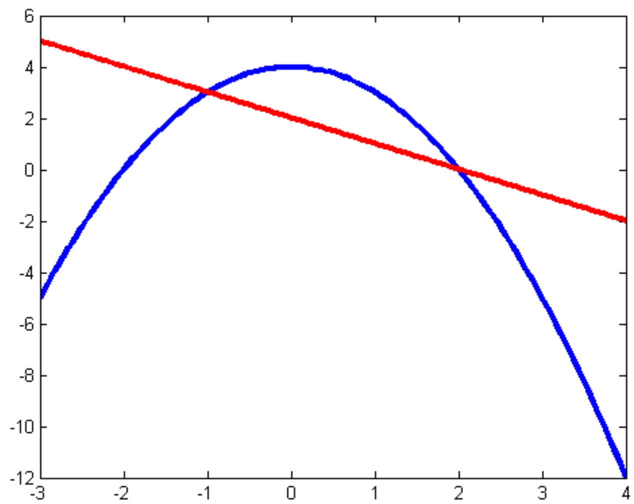
$$= \frac{8}{3} + 2 + \frac{1}{6} + \frac{8}{3} - 2 + \frac{1}{6} = \frac{16}{3} - 4 = \frac{17}{3}$$

QUESTION 5. Find the area of the region bounded by $y = 4 - x^2$ and $y = 2 - x$ over the interval $[-2, 3]$.

[3] In order to compute the area between the two curves we must first determine all the intersection points between the curves over the given interval $[-2, 3]$. We have

$$4 - x^2 = 2 - x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2,\ x = -1.$$

Next, we subdivide the given interval into subintervals separated by the intersection points determined above. Thus $[-2, -1] \cup [-1, 2] \cup [2, 3]$. On each of the subintervals, we determine which of the two functions is greater. We can accomplish this by looking at a graphical representation of the scenario.



Observe that $2 - x \geq 4 - x^2$ on $[-2, -1]$, $4 - x^2 \geq 2 - x$ on $[-1, 2]$ and $2 - x \geq 4 - x^2$ on $[2, 3]$. We can now set up the integrals that compute the desired area:

$$\int_{-2}^{-1} \Big(2 - x - (4 - x^2)\Big)\, dx + \int_{-1}^{2} \Big(4 - x^2 - (2 - x)\Big)\, dx + \int_{2}^{3} \Big(2 - x - (4 - x^2)\Big)\, dx$$

$$\int_{-2}^{-1} \Big(x^2 - x - 2\Big)\, dx + \int_{-1}^{2} \Big(2 + x - x^2\Big)\, dx + \int_{2}^{3} \Big(x^2 - x - 2\Big)\, dx$$

$$\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-2}^{-1} + \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-1}^{2} + \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{2}^{3}$$

$$\frac{11}{6} + \frac{9}{2} + \frac{11}{6} = \frac{11}{3} + \frac{9}{2} = \frac{49}{6}.$$

Question 6.

(a) Use finite approximations to estimate the area under the function $f(x) = \ln(x+1)$ between $x = 0$ and $x = 1$ using a righthand sum with four rectangles of equal width.

First, we calculate the width of each rectangle: $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Thus, the right end points are $x_1 = 0.25$, $x_2 = 0.50$, $x_3 = 0.75$ and $x_4 = 1$. The righthand estimate with four rectangles, R_4, is given by

$$R_4 = \Delta x \sum_{i=1}^{4} f(x_i) = 0.25\Big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\Big)$$
$$= 0.25\Big(\ln(1.25) + \ln(1.5) + \ln(1.75) + \ln(2)\Big) \approx 0.4703.$$

(b) Use finite approximations to estimate the area under the function $f(x) = \ln(x+1)$ between $x = 0$ and $x = 1$ using a lefthand sum with four rectangles of equal width.

The rectangle width is again given by $\Delta x = \frac{1}{4}$ and the left end points are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.50$, and $x_3 = 0.75$. The lefthand estimate with four rectangles, L_4, is given by

$$L_4 = \Delta x \sum_{i=0}^{3} f(x_i) = 0.25\Big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\Big)$$
$$= 0.25\Big(\ln(1) + \ln(1.25) + \ln(1.50) + \ln(1.75)\Big) \approx 0.2971.$$

(c) Calculate $\displaystyle\int_0^1 \ln(x+1)\,dx$.

The calculation of this integral requires integration by parts. We let $u = \ln(x+1)$ and $dv = dx$, then $du = \frac{1}{x+1}dx$ and $v = x$. Thus

$$\int_0^1 \ln(x+1)\,dx = x\ln(x+1)\Big|_0^1 - \int_0^1 \frac{x}{x+1}\,dx.$$

The integral on the righthand side of the above expression requires substitution: we let $w = x+1$, $dw = dx$ and we also note that $x = w - 1$. We also change the limits of integration: when $x = 0$, $w = 1$ and when $x = 1$, $w = 2$. Substituting these quantities into the integral we obtain

$$\int_1^2 \frac{w-1}{w}\,dw = \int_1^2 1 - \frac{1}{w}\,dw = w - \ln(w)\Big|_1^2 = 1 - \ln(2) \approx 0.3069$$

Going back to the original integral we obtain

$$\int_0^1 \ln(x+1)\,dx = x\ln(x+1)\Big|_0^1 - \int_0^1 \frac{x}{x+1} = \ln(2) - 0 - 0.3069 = 0.3862.$$

(d) Compare your answers in (a), (b) and (c). Which Riemann sum underestimates the actual value of $\int_0^1 \ln(x+1)\,dx$, and which Riemann sum overestimates it? Why? (A short explanation is enough)

We see that R_4 overestimates the actual value of $\int_0^1 \ln(x+1)\,dx$, whereas L_4 underestimates the actual value. This is because the function we are integrating, $\ln(x+1)$, is an increasing function.