

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	206/2	All

Examination	Date	Time	Pages
Final	December 2013	3 Hours	2

Instructors	Course Examiner
A. Bellahnid, D. Gernazian, D. Pearce, T. Hughes, U. Mgbemena	D. Sen

Special Instructions

▷ Only approved calculators are allowed.

MARKS

[4] 1. Simplify the expressions below. Do not use a calculator.

(a) $-\sqrt{18} + 2\sqrt{8} + \sqrt{72}$ (b) $\frac{2}{3} \ln 8 - \ln(5^2 - 1)$

[4] 2. Rationalize the denominator:

(a) $\frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}}$ (b) $\frac{2 - \sqrt{5}}{2 + 3\sqrt{5}}$

[6] 3. Simplify the expressions:

(a) $4x^2(4x^3 - 3x^2 - 2x) - 5x(4x^4 - 3x^3 + 2x^2 + 4x)$ (b) $\frac{x^2 + 4x + 4}{x^2 + 3x + 2}$

[8] 4. Factor the polynomials completely:

(a) $6x^2 + 5x + 1$ (b) $8x^3 + 27$

[4] 5. Perform the arithmetic operations and simplify:

$$\frac{x}{x-3} - \frac{x+1}{x^2+5x-24}$$

[9] 6. Solve the equations:

(a) $\frac{x}{x^2-9} + \frac{4}{x+3} = \frac{3}{x^2-9}$ (b) $\log_5(x+3) = 1 - \log_5(x-1)$

(c) $8^{-x+14} = (16)^x$

[8] 7. Solve the inequalities, express your answer using set notation or interval notation:

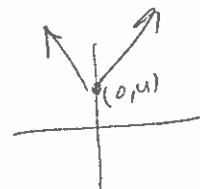
(a) $0 < \frac{3x+2}{2} < 4$ (b) $\left| \frac{2x+3}{3} - \frac{1}{2} \right| < 1$

- [4] 8. Solve the system of equations:

$$\begin{aligned}x^2 + y^2 &= 36 \\x + y &= 8\end{aligned}$$

Note: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- [8] 9. (a) Which of the points $A(2, 3)$, $B(6, 4)$ is closer to the point $C(4, 3)$?
 (b) Show that the equation $2x^2 + 2y^2 + 8x + 7 = 0$ represents a circle. Find coordinates of the center and radius of the circle.



- [6] 10. Find the domain and range of the functions (do not graph):

(a) $f(x) = \frac{3x}{x^2 - 4}$ (b) $g(x) = \sqrt{-x - 2}$ (c) $h(x) = |x| + 4$

- [5] 11. Sketch the graph of the function $f(x) = \log(x - 4) + 2$, starting from the graph of the function $g(x) = \log x$ and using appropriate transformations.

$$\frac{2}{|x|}$$

- [8] 12. Let $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{-4}{x}$. Find:

(a) fg (b) $\frac{f}{g}$ (c) $f \circ g$ (d) $g \circ f$

- [8] 13. (a) Find the inverse of the function $f(x) = \frac{2x}{3x-1}$.

(b) Find the vertical and horizontal asymptotes of both f and f^{-1} above.

- [5] 14. The diagonal of a rectangle measures 10 inches. If the length is 2 inches more than the width, find the dimensions of the rectangle.

- [5] 15. From each corner of a square piece of sheet metal, remove a square of side 9 centimeters. Turn up the edges to form an open box. If the box is to hold 144 cubic centimeters, what should be the dimensions of the piece of sheet metal?

- [8] 16. The number N of bacteria present in a culture at time t (in hours) obeys the law of uninhibited growth

$$N(t) = 1000e^{0.01t}$$

- (a) Determine the number of bacteria at $t = 0$ hours.
 (b) What is the population after 4 hours?
 (c) When will the number of bacteria double?

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CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Section(s)	
Mathematics	206/4	All	
Examination	Date	Time	Pages
Final	April 2013	3 Hours	2
Instructors	A. Bradley, M. Padamadan, R. Mearns, V. Kalvin		Course Examiner
			D. Sen
Special Instructions			
▷ Only approved calculators are allowed.			

MARKS

- [4] 1. Simplify the expressions below. Do not use a calculator.

(a) $3\sqrt{5} + \sqrt{20} - 2\sqrt{45}$ (b) $\log_5 10 + \log_5 (3^3 - 12) - \log_5 30$

- [4] 2. Rationalize the denominator:

(a) $\frac{\sqrt{5}}{3 - \sqrt{2}}$ (b) $\frac{1 + \sqrt{2}}{1 - \sqrt{2}}$

- [6] 3. Simplify the expressions:

(a) $6x(x^3 - x^2 - 3x) - 4x(3x^4 - 2x^3 + 3x^2 + x)$ (b) $\frac{4x^2 - 8x}{12x - 24}$

- [8] 4. Factor the polynomials completely:

(a) $2x^2 - x - 6$ (b) $x^7 - x^5$

- [4] 5. Perform the arithmetic operations and simplify:

$$\frac{x+4}{x^2-x-2} - \frac{2x+3}{x^2+2x-8}$$

- [9] 6. Solve the equations:

(a) $\frac{x+1}{x^2+2x} - \frac{x+4}{x^2+x} = \frac{-3}{x^2+3x+2}$ (b) $3(169^x) = 39$

(c) $\log_4(5 - 2x) = -2$

- [8] 7. Solve the inequalities, express your answer using set notation or interval notation:

(a) $1 \leq \frac{-2-3x}{7} < 4$ (b) $2|x+1| - 3 \leq 7$

- [4] 8. Solve the system of equations:

$$\begin{aligned}2x^2 + y^2 &= 1 \\ 2x - y &= -1\end{aligned}$$

- [8] 9. (a) Which of the points $A(9, 3)$, $B(4, 5)$ is closer to the point $C(13, 4)$?
(b) Show that the equation $x^2 + y^2 - 8x + 20y + 107 = 0$ represents a circle. Find coordinates of the center and radius of the circle.

- [6] 10. Find the domain and range of the functions (do not graph):

$$(a) f(x) = \frac{3}{x^2 + 9} \quad (b) g(x) = \sqrt{4 - 3x} \quad (c) h(x) = |x - 4|$$

- [5] 11. Sketch the graph of the function $f(x) = -2(x - 3)^2$, starting from the graph of the function $g(x) = x^2$ and using appropriate transformations.

- [8] 12. Let $f(x) = \frac{x + 3}{2x - 4}$ and $g(x) = \frac{3x - 1}{x - 2}$. Find:

$$(a) fg \quad (b) \frac{f}{g} \quad (c) f \circ g \quad (d) g \circ f$$

- [8] 13. (a) Find the inverse of the function $f(x) = \frac{2x + 5}{x + 7}$.

(b) Find the vertical and horizontal asymptotes of both f and f^{-1} above.

- [5] 14. Connie can clean her house in 2 hours. If Alvaro helps her, they can clean the house in 1 hour and 15 minutes together. How long would it take Alvaro to clean the house by himself?

- [5] 15. A total of \$18,000 is invested, some in stocks and some in bonds. If the amount invested in bonds is half that invested in stocks, how much is invested in each category?

- [8] 16. The logistic growth model

$$P(t) = \frac{1000}{1 + 32.33e^{-0.439t}}$$

represents the population (in grams) of a bacterium after t hours.

- (a) What is the population after 9 hours?
(b) When will the population be 700 grams?
(c) How long does it take for the population to reach one-half the carrying capacity?

December 2013 Final Exam Solutions

$$\begin{aligned} \textcircled{1} \text{ a) } & -\sqrt{18} + 2\sqrt{8} + \sqrt{72} \\ & -\sqrt{9 \cdot 2} + 2\sqrt{4 \cdot 2} + \sqrt{8 \cdot 9} \\ & -3\sqrt{2} + 4\sqrt{2} + 3\sqrt{2 \cdot 4} \\ & \quad \quad \quad \sqrt{2} + 6\sqrt{2} \\ & \quad \quad \quad 7\sqrt{2} \end{aligned}$$

$$\text{b) } \frac{2}{3} \ln 8 - \ln(5^2 - 1)$$

$$\frac{2}{3} \ln 8 - \ln(25 - 1)$$

$$\frac{2}{3} \ln 8 - \ln 24$$

$$\ln(8)^{2/3} - \ln 24$$

$$\ln 4 - \ln 24$$

$$\ln\left(\frac{4}{24}\right)$$

$$\ln\left(\frac{1}{6}\right)$$

BEDMAS
 deal with brackets first!
 Brackets
 Exponents
 Division ←
 Multiplication
 Addition ←
 Subtraction

2

$$\text{a) } \frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}} \cdot \frac{\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + 3\sqrt{2}}$$

$$\frac{\sqrt{2}(\sqrt{3} + 3\sqrt{2})}{3 - 9(2)}$$

$$\frac{\sqrt{6} + 3(2)}{-15}$$

$$\frac{6 + \sqrt{6}}{-15}$$

$$-\frac{2}{5} - \frac{\sqrt{6}}{15}$$

K. Kefauver

2)

$$b) \frac{2 - \sqrt{5}}{2 + 3\sqrt{5}} \cdot \frac{2 - 3\sqrt{5}}{2 - 3\sqrt{5}}$$

$$\frac{(2 - \sqrt{5})(2 - 3\sqrt{5})}{4 - 9(5)}$$

$$\frac{4 - 6\sqrt{5} - 2\sqrt{5} + 3\sqrt{25}}{-41}$$

$$\frac{4 - 8\sqrt{5} + 3(5)}{-41}$$

$$\frac{19 - 8\sqrt{5}}{-41}$$

3

$$a) 4x^2(4x^3 - 3x^2 - 2x) - 5x(4x^4 - 3x^3 + 2x^2 + 4x)$$

$$16x^5 - 12x^4 - 8x^3 - 20x^5 + 15x^4 - 10x^3 - 20x^2$$

$$-4x^5 + 3x^4 - 18x^3 - 20x^2$$

$$b) \frac{x^2 + 4x + 4}{x^2 + 3x + 2}$$

product	1, 4	(2, 2)
sum	5	4

product	1, 2
sum	3

thus,

factor numerator:

$$x^2 + 4x + 4$$

$$x^2 + 2x + 2x + 4$$

$$x(x+2) + 2(x+2)$$

$$(x+2)(x+2)$$

factor denominator:

$$x^2 + 3x + 2$$

$$x^2 + 2x + x + 2$$

$$x(x+2) + (x+2)$$

$$(x+2)(x+1)$$

$$\frac{(x+2)(x+2)}{(x+2)(x+1)} = \frac{x+2}{x+1} \quad \text{where } x \neq -2$$

$$x \neq -1$$

Khefala

④ a) $6x^2 + 5x + 1$

check: is it prime? $b^2 - 4ac = 5^2 - 4(6)(1) = 1$
 Since $b^2 - 4ac = \overset{\text{positive}}{\text{value}}$ (not 0) there is real solutions
can be factored

product	6, 1	3, 2
sum	7	5

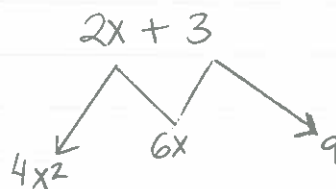
$$6x^2 + 5x + 1$$

$$6x^2 + 3x + 2x + 1$$

$$3x(2x + 1) + (2x + 1)$$

$$\boxed{(2x + 1)(3x + 1)}$$

b) $8x^3 + 27$
 $2^3x^3 + 3^3$
 $(2x)^3 + 3^3$ *this is a sum of cubes*
 $\boxed{(2x + 3)(4x^2 - 6x + 9)}$



can this be factored further?

$$(2x + 3)(4x^2 - 6x + 9)$$

$$= 8x^3 - 12x^2 + 18x + 12x^2 - 18x + 27$$

$$= 8x^3 + 27$$

No, because:
 $b^2 - 4ac \rightarrow (-6)^2 - 4(4)(9) = -108$

since $b^2 - 4ac < 0$ this trinomial is prime
 (can't be factored further)

⑤ $\frac{x}{x-3} - \frac{x+1}{x^2 + 5x - 24}$

product	6, -4	-3, 12	3, -12	3, +8
sum	2	10	-10	5

$$\frac{x}{(x-3)} - \frac{x+1}{(x-3)(x+8)}$$

$$\frac{x(x+8) - (x+1)}{(x-3)(x+8)}$$

$$x^2 + 5x - 24$$

$$x^2 + 8x - 3x - 24$$

$$x(x+8) - 3(x+8)$$

$$(x+8)(x-3)$$

$$\frac{x^2 + 8x - x - 1}{(x-3)(x+8)} = \frac{x^2 + 7x - 1}{(x-3)(x+8)}$$

K. Koles

6

$$a) \frac{x}{x^2-9} + \frac{4}{x+3} = \frac{3}{x^2-9}$$

where $x \neq -3$
 $x \neq 3$

$$\frac{x}{(x-3)(x+3)} + \frac{4}{x+3} = \frac{3}{(x-3)(x+3)} \quad \left. \vphantom{\frac{x}{(x-3)(x+3)}} \right\} \text{multiply the entire expression by } (x-3)(x+3)$$

$$x + 4(x-3) = 3$$

$$x + 4x - 12 = 3$$

$$5x = 15$$

$$x = 3$$

But $x \neq -3, x \neq 3$; so No solution.

$$b) \log_5(x+3) = 1 - \log_5(x-1)$$

The domain is $x+3 > 0$ $x-1 > 0$
 $x > -3$ $x > 1$

so our solution must satisfy $x > 1$

$$\log_5(x+3) = 1 - \log_5(x-1)$$

$$\log_5(x+3) + \log_5(x-1) = 1$$

$$\log_5(x+3)(x-1) = 1$$

$$5^1 = (x+3)(x-1)$$

$$5 = x^2 - x + 3x - 3$$

$$0 = x^2 + 2x - 8$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \frac{-2 \pm \sqrt{4 - 4(1)(-8)}}{2} = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2}$$

$$x = -4, 2$$

Since $x > 1$, our solution is $x = 2$.

K. Kefeler

6)

$$\begin{aligned} c) \quad & 8^{-x+14} = 16^x \\ & (2^3)^{-x+14} = (2^4)^x \\ & (2)^{-3x+42} = (2)^{4x} \end{aligned}$$

hence, $-3x + 42 = 4x$

$$42 = 7x$$

$$x = 6$$

check: $8^{-6+14} = 16^6$

$$8^8 = 16^6$$

$$16777216 = 16777216 \quad \text{OK}$$

7) a) $0 < \frac{3x+2}{2} < 4$

$$0 < \frac{3x+2}{2}$$

$$0 < 3x+2$$

$$-2 < 3x$$

$$-\frac{2}{3} < x$$

$$\frac{3x+2}{2} < 4$$

$$3x+2 < 8$$

$$3x < 6$$

$$x < 2$$

thus

$$-\frac{2}{3} < x < 2$$

b) $\left| \frac{2x+3}{3} - \frac{1}{2} \right| < 1$

$$\text{thus, } -1 < x < \frac{3}{4}$$

① $+\left(\frac{2x+3}{3} - \frac{1}{2}\right) < 1$

$$2(2x+3) - 3 < 6$$

$$4x + 6 - 3 < 6$$

$$4x < 3 \quad \text{so } x < \frac{3}{4}$$

② $-\left(\frac{2x+3}{3} - \frac{1}{2}\right) < 1$

$$\left(\frac{2x+3}{3} - \frac{1}{2}\right) > -1$$

$$2(2x+3) - 3 > -1$$

$$4x + 3 > -1 \rightarrow x > -1$$

K. Kefeler

8) Solve

$$x^2 + y^2 = 36$$

$$x + y = 8 \rightarrow y = 8 - x$$

$$x^2 + (8-x)^2 = 36$$

$$x^2 + (8-x)(8-x) = 36$$

$$x^2 + 64 - 16x + x^2 = 36$$

$$2x^2 - 16x + 28 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{16 \pm \sqrt{(-16)^2 - 4(2)(28)}}{4}$$

$$x = \frac{16 \pm \sqrt{32}}{4} = 4 \pm \frac{\sqrt{8 \cdot 4}}{4}$$

$$x = 4 \pm \frac{2\sqrt{8}}{4}$$

$$\sqrt{8} = \sqrt{2 \cdot 4}$$

$$= 2\sqrt{2}$$

$$x = 4 \pm \frac{4\sqrt{2}}{4} = 4 \pm \sqrt{2}$$

Solutions

If $x = 4 + \sqrt{2}$ then $y = 8 - x = 8 - 4 - \sqrt{2} = 4 - \sqrt{2}$

If $x = 4 - \sqrt{2}$ then $y = 8 - x = 8 - 4 + \sqrt{2} = 4 + \sqrt{2}$

Check $x^2 + y^2 = 36$
 $x + y = 8$

• for $x = 4 + \sqrt{2}$; $y = 4 - \sqrt{2}$

$$x^2 + y^2 = 36$$

$$(4 + \sqrt{2})^2 + (4 - \sqrt{2})^2 = 36$$

$$x + y = 8$$

$$4 + \sqrt{2} + 4 - \sqrt{2} = 8 \quad \text{OK}$$

• for $x = 4 - \sqrt{2}$; $y = 4 + \sqrt{2}$

$$x^2 + y^2 = 36$$

$$\text{OK } (4 - \sqrt{2})^2 + (4 + \sqrt{2})^2 = 36$$

$$x + y = 8$$

$$4 - \sqrt{2} + 4 + \sqrt{2} = 8 \quad \text{OK}$$

9)

a) distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

for point A (2,3)

point C (4,3)

$$\text{distance} = \sqrt{(2-4)^2 + (3-3)^2} = \underline{\underline{2}}$$

for point B (6,4)

point C (4,3)

$$\text{distance} = \sqrt{(6-4)^2 + (4-3)^2} = \sqrt{5} \approx 2.2$$

therefore, point A is closer to point C.

b) $2x^2 + 2y^2 + 8x + 7 = 0$

$$\underbrace{2x^2 + 8x} + 2y^2 = -7$$

complete the square

$$2(x^2 + 4x) + 2y^2 = -7$$

$$2(x^2 + 4x + 4) + 2y^2 = -7 + 8$$

$$2(x+2)^2 + 2y^2 = 1$$

$$2(x+2)^2 + 2(y-0)^2 = 1$$

$$(x+2)^2 + (y-0)^2 = 1/2$$

← circle must take the form:
 $(x-h)^2 + (y-k)^2 = r^2$
 $(x-(-2))^2 + (y-0)^2 = 1/2$

→ center (h, k) = (-2, 0)

→ Since $r^2 = 1/2$, then $r = \sqrt{1/2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

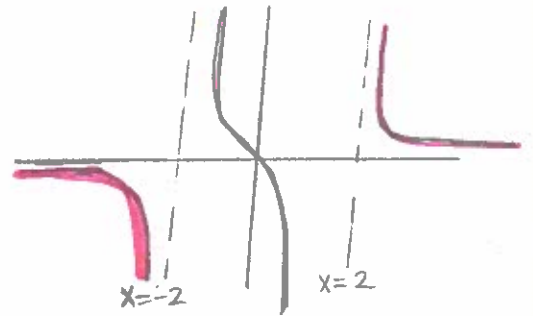
10) a) (you can graph to solve it, but no marks given for graph)

$$f(x) = \frac{3x}{x^2-4} = \frac{3x}{(x-2)(x+2)}$$

$$\text{domain: } \{x \mid x \neq -2, x \neq 2\}$$

$$\text{Range: } y \in \mathbb{R} \text{ (all real numbers)}$$

$$y \in (-\infty, \infty)$$



* to find the range, best to graph it

• this problem you need to graph $f(x) = \frac{3x}{x^2-4}$ } 8 step procedure or do a rough sketch

$$\text{b) } g(x) = \sqrt{-x-2}$$

$$g(x) = \sqrt{-(x+2)}$$

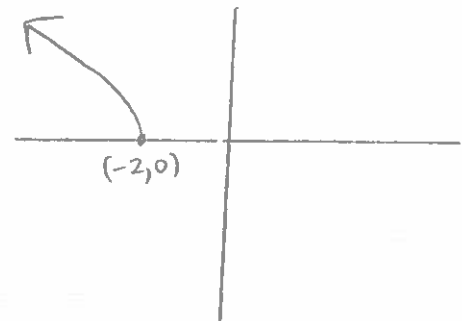
$$\text{for domain, } -x-2 \geq 0$$

$$-x \geq 2$$

$$x \leq -2$$

$$\text{so } x \in \mathbb{R} \text{ from } (-\infty, -2]$$

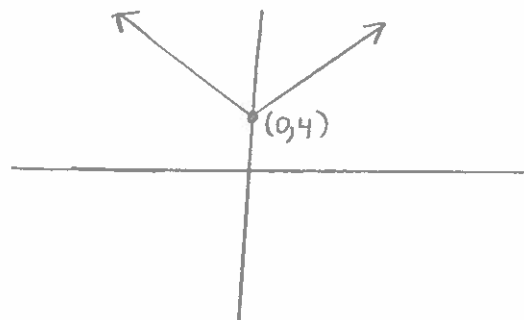
$$\text{Range: } y \in \mathbb{R} \text{ from } [0, \infty)$$



$$\text{c) } h(x) = |x| + 4$$

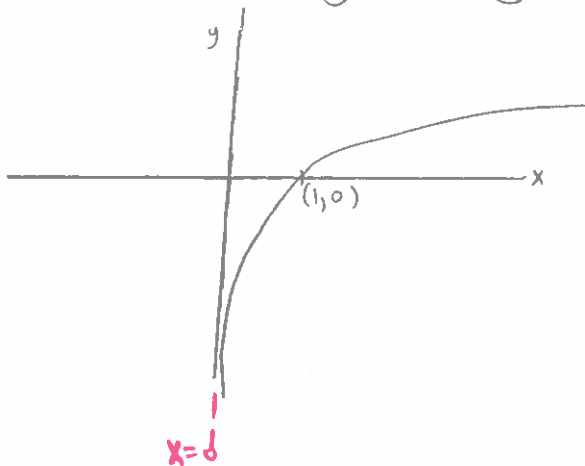
$$\text{domain: } x \in \mathbb{R} \text{ } (-\infty, \infty)$$

$$\text{range: } y \in \mathbb{R} \text{ } [4, \infty)$$



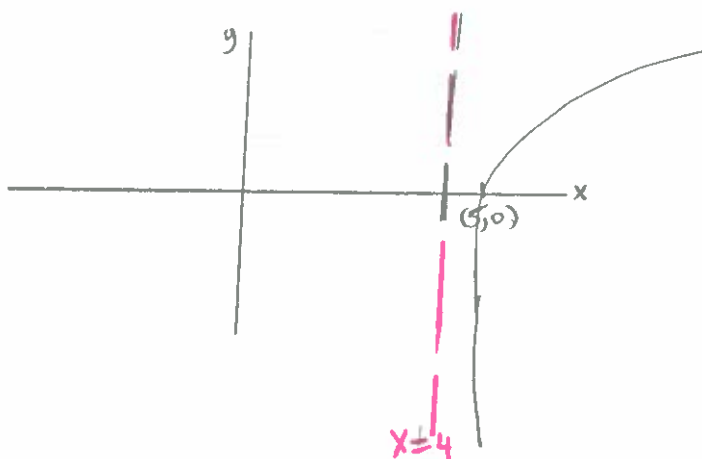
K. Kepler

11) start with $g(x) = \log x$



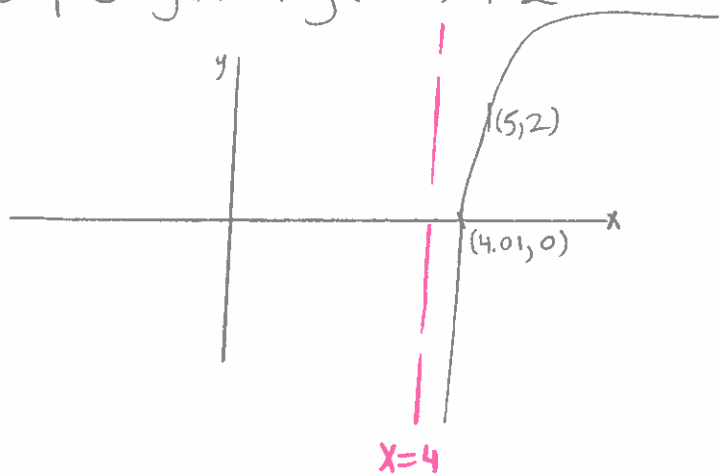
asymptote at $x=0$
domain: $x > 0$

step 2: $g(x) = \log(x-4)$



shifts 4 units to the right
 $x=4$ is a vertical
asymptote
 $\log(4-4) = \log(0)$
 \hookrightarrow undefined

step 3: $g(x) = \log(x-4) + 2$



shifts up 2 units
vertically
 $(5, 0) \rightarrow$ becomes $(5, 2)$

K. Kofler

⑫ let $f(x) = \frac{x}{x-1}$

$$g(x) = \frac{-4}{x}$$

- find
- a) $fg \rightarrow f(x)g(x)$
 - b) $f/g \rightarrow f(x)/g(x)$
 - c) $f \circ g \rightarrow f(g(x))$
 - d) $g \circ f \rightarrow g(f(x))$

a) $f(x)g(x) = \left(\frac{x}{x-1}\right)\left(\frac{-4}{x}\right) = \frac{-4}{x-1}$

domain: $\{x \mid x \neq 0, x \neq 1\}$

b) $\frac{f}{g} = \frac{f(x)}{g(x)} = \frac{x}{x-1} \div \frac{-4}{x} = \frac{x^2}{-4(x-1)}$

domain: $\{x \mid x \neq 0, x \neq 1\}$

c) $f \circ g = f(g(x))$
 $f\left(\frac{-4}{x}\right)$

$$\frac{\frac{-4}{x}}{\frac{-4}{x} - 1} = \frac{-4}{x} \div \frac{-4-x}{-1}$$

$$= \frac{-4}{x} \cdot \frac{-1}{-4-x} = \frac{4}{(x)(-4-x)}$$

$f \circ g$'s
 domain: $\{x \mid x \neq -4, x \neq 0\}$

d) $g(f(x)) \rightarrow g\left(\frac{x}{x-1}\right)$

$$\frac{-4}{\left(\frac{x}{x-1}\right)} = \frac{-4(x-1)}{x} = \frac{-4x+4}{x} = -4 + \frac{4}{x}$$

domain: $\{x \mid x \neq 0, x \neq 1\}$

$x \neq 0$
 $-4-x \neq 0$
 $-4 \neq x$

K. K. K.

13 $f(x) = \frac{2x}{3x-1}$

a) $x = \frac{2y}{3y-1}$

$x(3y-1) = 2y$
 $3xy - x = 2y$
 $3xy - 2y = x$
 $y(3x-2) = x$

$f^{-1}(x) = y = \frac{x}{3x-2}$

verify that ① $f(f^{-1}(x)) = x$ and ② $f^{-1}(f(x)) = x$:

① $f(f^{-1}(x)) = f\left(\frac{x}{3x-2}\right) = \frac{2\left(\frac{x}{3x-2}\right)}{3\left(\frac{x}{3x-2}\right)-1} = \frac{\frac{2x}{3x-2}}{\frac{3x-(3x-2)}{3x-2}}$
 $= \frac{2x}{3x-2} \div \frac{2}{3x-2}$
 $= \frac{2x}{2}$
 $= x$

domain of $f \circ f^{-1}$:
 $\{x \mid x \neq 2/3\}$

② $f^{-1}(f(x)) = f^{-1}\left(\frac{2x}{3x-1}\right) = \frac{\frac{2x}{3x-1}}{3\left(\frac{2x}{3x-1}\right)-2} = \frac{\frac{2x}{3x-1}}{\frac{6x-2(3x-1)}{3x-1}} = \frac{2x/(3x-1)}{\frac{6x-6x+2}{3x-1}}$
 $= \frac{2x}{3x-1} \div \frac{2}{3x-1}$

domain of $f^{-1} \circ f$: $\{x \mid x \neq 1/3\}$

$= \frac{2x}{2}$

thus, verification is successful meaning $f^{-1}(x)$ solved $= x$

K. Kefeler

b) $f(x) = \frac{2x}{3x-1}$ vertical asymptote $x = 1/3$

$f^{-1}(x) = \frac{x}{3x-2}$ vertical asymptote $x = 2/3$

since improper, use long division:

$$3x-1 \overline{) \begin{array}{r} \frac{2}{3} \\ 2x \\ -(2x - 2/3) \\ \hline 2/3 \end{array}}$$

$$\frac{2}{3} + \frac{2/3}{(3x-1)} = \frac{2x}{3x-1} = f(x)$$

so as $x \rightarrow \infty$ $f(x) = \frac{2}{3} + \frac{2/3}{(3x-1)}$

$$f(x) = 2/3$$

so horizontal asymptote at $y = 2/3$ for $f(x)$

• since improper, use long division:

$$3x-2 \overline{) \begin{array}{r} \frac{1}{3} \\ x \\ -(x - 2/3) \\ \hline 2/3 \end{array}}$$

so $\frac{x}{3x-2} = \frac{1}{3} + \frac{2/3}{3x-2} = f^{-1}(x)$

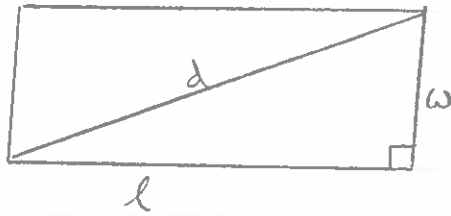
as $x \rightarrow \infty$ $\frac{2/3}{3x-2} \approx \frac{2/3}{3x} \approx 0$

so $f^{-1}(x) = \frac{1}{3} + \frac{2/3}{3x-2} = 1/3$

so horizontal asymptote at $y = 1/3$ for $f^{-1}(x)$

K. Kepler

14)



$$d = 10 \text{ inches}$$

$$l = w + 2$$

Pythagoras' theorem: $a^2 + b^2 = c^2$

$$l^2 + w^2 = d^2$$

$$(w+2)^2 + w^2 = 10^2$$

$$w^2 + 4w + 4 + w^2 = 10^2$$

$$2w^2 + 4w + 4 = 10^2$$

$$2w^2 + 4w - 96 = 0$$

$$l = w + 2$$

$$d = 10$$

$$\begin{array}{l} (w+2)(w+2) \\ w^2 + 4w + 4 \end{array}$$

$$w = \frac{-4 \pm \sqrt{4^2 - 4(2)(-96)}}{2(2)} = \frac{-4 \pm 28}{4}$$

$$w = \cancel{-8}, 6$$

dimensions of rectangle must have positive value

$$w = 6$$

$$l = w + 2 = 8$$

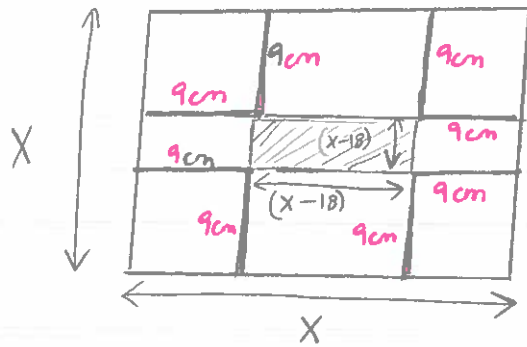
check $6^2 + 8^2 = 10^2$

$$100 = 100 \text{ OK}$$

check our length is 2 units more than our width (necessary requirement stated in problem)

K. Rajan

15)



shaded square
 $(X-18)(X-18) = \text{Area of base of box}$

$$\begin{aligned} \text{the volume of the box} &= (\text{Area of base}) (\text{height}) \\ 144 &= (X-18)(X-18)(9\text{cm}) \\ 144 &= 9(X^2 - 36X + 324) \\ 144 &= 9X^2 - 324X + 2916 \\ 0 &= 9X^2 - 324X + 2772 \end{aligned}$$

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{324 \pm \sqrt{5184}}{18}$$

$$X = \frac{324 \pm 72}{18} = 14, 22$$

$X \neq 14$ because the dimensions of the base of the box are $(X-18)(X-18)$

$$\text{So } X = 22 \text{ (must be!)}$$

$$X-18 = 22-18 = 4$$

check Volume of box = $4 \times 4 \times 9 = 144 \text{ cm}^3$ OK

$$16) N(t) = 1000 e^{0.01t}$$

t : hours

N : bacteria number

a) $t = 0$

$$N(t) = 1000 e^{0.01(0)} = 1000 e^0 = 1000$$

$$e^0 = 1$$

b) $N(t) = ?$ when $t = 4$ hours

$$N(t) = 1000 e^{0.01(4)} = 1040.81$$

c) what value of t for $N(t) = 2(1000) = 2000$

$$2000 = 1000 e^{0.01t}$$

$$2 = e^{0.01t}$$

$$\ln 2 = \ln e^{0.01t}$$

$$\ln 2 = 0.01t (\ln e)$$

$$\ln 2 = 0.01t$$

$$\frac{\ln 2}{0.01} = t$$

$$69.3 = t$$

fun fact:
 $\ln e = 1$

So the amount of bacteria will double at 69.3 hours.

K. Kefalos

April 2013 Final Exam Solutions

$$\textcircled{1} \text{ a) } \begin{aligned} & 3\sqrt{5} + \sqrt{20} - 2\sqrt{45} \\ & 3\sqrt{5} + \sqrt{5 \cdot 4} - 2\sqrt{9 \cdot 5} \\ & 3\sqrt{5} + 2\sqrt{5} - 6\sqrt{5} \\ & \quad - \sqrt{5} \end{aligned}$$

$$\text{b) } \begin{aligned} & \log_5 10 + \log_5 (3^3 - 12) - \log_5 30 \\ & \log_5 10 + \log_5 (27 - 12) - \log_5 30 \\ & \log_5 10 + \log_5 15 - \log_5 30 \\ & \log_5 \left[\frac{10(15)}{30} \right] \end{aligned}$$

$$\log_5 5 \longleftrightarrow 5^x = 5$$

$$1 \quad \text{on calculator: } \frac{\log 5}{\log 5} = 1$$

from change-of-base formula

$$\textcircled{2} \text{ a) } \frac{\sqrt{5}}{3 - \sqrt{2}} \cdot \frac{3 + \sqrt{2}}{3 + \sqrt{2}}$$

$$\frac{3\sqrt{5} + \sqrt{10}}{9 - 2}$$

$$\frac{3\sqrt{5} + \sqrt{10}}{7}$$

$$\text{b) } \frac{1 + \sqrt{2}}{1 - \sqrt{2}} \cdot \frac{1 + \sqrt{2}}{1 + \sqrt{2}}$$

$$= \frac{1 + \sqrt{2} + \sqrt{2} + 2}{1 - 2}$$

$$= \frac{1 + 2\sqrt{2} + 2}{-1} = \frac{3 + 2\sqrt{2}}{-1} = -3 - 2\sqrt{2}$$

K. Kaplan

③ a) $6x(x^3 - x^2 - 3x) - 4x(3x^4 - 2x^3 + 3x^2 + x)$
 $\underline{6x^4} - 6x^3 - 18x^2 - 12x^5 + \underline{8x^4} - 12x^3 - 4x^2$
 $-12x^5 + 14x^4 - 18x^3 - 22x^2$

b) $\frac{4x^2 - 8x}{12x - 24}$

$= \frac{4x(x/2)}{12(x/2)}$

$= \frac{x}{3}$ where $x \neq 2$

← don't forget
ALWAYS STATE your
 RESTRICTIONS

④ a) $2x^2 - x - 6$
 $2x^2 + 3x - 4x - 6$
 $x(2x+3) - 2(2x+3)$
 $(2x+3)(x-2)$

product	-3, 4	<u>3, -4</u>	6, -2	-6, 2	-12, 1
sum	1	<u>-1</u>	4	-4	-11

b) $x^7 - x^5$
 $x^5(x^2 - 1)$
 $x^5(x-1)(x+1)$

⑤ $\frac{x+4}{x^2 - x - 2} - \frac{2x+3}{x^2 + 2x - 8}$

$x^2 - x - 2$
 $x^2 - 2x + x - 2$
 $x(x-2) + (x-2)$
 $(x-2)(x+1)$

$= \frac{x+4}{(x-2)(x+1)} - \frac{2x+3}{(x+4)(x-2)}$

$x^2 + 2x - 8$
 $x^2 + 4x - 2x - 8$
 $x(x+4) - 2(x+4)$
 $(x+4)(x-2)$

$= \frac{(x+4)(x+4) - (2x+3)(x+1)}{(x-2)(x+1)(x+4)}$

$= \frac{x^2 + 8x + 16 - (2x^2 + 2x + 3x + 3)}{(x-2)(x+1)(x+4)} = \frac{-x^2 + 3x + 13}{(x-2)(x+1)(x+4)}$

$$6) \quad a) \quad \frac{x+1}{x^2+2x} - \frac{x+4}{x^2+x} = \frac{-3}{x^2+3x+2}$$

$$\frac{x+1}{x(x+2)} - \frac{x+4}{x(x+1)} = \frac{-3}{(x+2)(x+1)}$$

multiply everything by $(x)(x+2)(x+1)$

$$\cancel{(x)}\cancel{(x+2)}\cancel{(x+1)} \frac{(x+1)}{\cancel{x}\cancel{(x+2)}} - \cancel{(x)}\cancel{(x+2)}\cancel{(x+1)} \frac{(x+4)}{\cancel{x}\cancel{(x+1)}} = \frac{-3\cancel{(x)}\cancel{(x+2)}\cancel{(x+1)}}{\cancel{(x+2)}\cancel{(x+1)}}$$

$$(x+1)(x+1) - (x+2)(x+4) = -3x$$

$$x^2 + 2x + 1 - (x^2 + 6x + 8) = -3x$$

$$x^2 + 2x + 1 - x^2 - 6x - 8 = -3x$$

$$-4x - 7 = -3x$$

$$-x = 7$$

$$x = -7$$

check: $\frac{x+1}{x^2+2x} - \frac{x+4}{x^2+x} = \frac{-7+1}{49-14} - \frac{-3}{49-7} = -\frac{1}{10}$

$$\frac{-3}{x^2+3x+2} = \frac{-3}{49+3(-7)+2} = \frac{-1}{10} \quad \text{OK}$$

so $x = \{-7\}$ is a solution

b) $3(169^x) = 39$
 $169^x = 13$
 $(13^2)^x = 13$
 $13^{2x} = 13^1$

so $2x = 1$
 $x = 1/2$

Alternative solution:
 $169^x = 13$
 $\log 169^x = \log 13$
 $x(\log 169) = \log 13$
 $x = \frac{\log 13}{\log 169} = \frac{1}{2}$

check: $3(169^{1/2}) = 39 \quad \text{OK}$

so $x = \{1/2\}$ is a solution

K. Kepler

$$c) \log_4 (5-2x) = -2$$

$$4^{-2} = (5-2x)$$

$$\frac{1}{4^2} = 5-2x$$

$$\frac{1}{16} - 5 = -2x$$

$$\frac{-79}{16} = -2x$$

$$\frac{79}{32} = x$$

domain of solution must be: $5-2x > 0$

$$5 > 2x$$

$$\frac{5}{2} > x$$

$$2.5 > x$$

Since $x = \frac{79}{32} = 2.46875$ it satisfies our domain ($2.5 > x$)

check $\log_4 (5-2(2.46875)) = \log_4 0.0625 = \frac{\log 0.0625}{\log 4} = -2$

OK

$$⑦ a) 1 \leq \frac{-2-3x}{7} < 4$$

$$\begin{array}{l} \swarrow \quad \searrow \\ 1 \leq \frac{-2-3x}{7} \qquad \frac{-2-3x}{7} < 4 \end{array}$$

$$7 \leq -2-3x$$

$$9 \leq -3x$$

$$-3 \geq x$$

$$-2-3x < 28$$

$$-3x < 30$$

$$x > -10$$

thus, $-10 < x \leq -3$, or $(-10, -3]$

$$\textcircled{b) } 2|x+1| - 3 \leq 7$$

$$2|x+1| \leq 10$$

$$|x+1| \leq 5$$

$$\begin{array}{l} \swarrow \quad \searrow \\ (x+1) \leq 5 \quad \quad -(x+1) \leq 5 \\ x \leq 4 \quad \quad x+1 \geq -5 \\ x \leq 4 \quad \quad x \geq -6 \end{array}$$

thus, $-6 \leq x \leq 4$, OR $[-6, 4]$

$$\textcircled{8} \quad 2x^2 + y^2 = 1$$

$$2x - y = -1 \rightarrow 2x + 1 = y$$

$$2x^2 + (2x+1)^2 = 1$$

$$2x^2 + (2x+1)(2x+1) = 1$$

$$2x^2 + (4x^2 + 4x + 1) = 1$$

$$6x^2 + 4x + 1 = 1$$

$$6x^2 + 4x = 0$$

$$2x(3x+2) = 0$$

so $x=0$ OR $x = -2/3$

• If $x=0$, $y = 2x+1 \rightarrow 2(0)+1 \rightarrow y=1$

this solution is possible because:

$$2x^2 + y^2 = 1 \rightarrow 2(0)^2 + 1^2 = 1 \quad \underline{\text{OK}}$$

$$2x - y = -1 \rightarrow 2(0) - 1 = -1 \quad \underline{\text{OK}}$$

• If $x = -2/3$, $y = 2(-2/3) + 1 = -1/3$

$$2x^2 + y^2 = 1 \rightarrow 2(-2/3)^2 + (-1/3)^2 = 1 \quad \underline{\text{OK}}$$

$$2x - y = -1 \rightarrow 2(-2/3) - (-1/3) = -1 \quad \underline{\text{OK}}$$

So, solutions for (x, y) : $\{ (0, 1), (-2/3, -1/3) \}$

9) a) distance: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

for A (9,3) and C(13,4):

$$\text{distance} = \sqrt{(13-9)^2 + (4-3)^2} = \sqrt{17} \approx 4.123$$

for B (4,5) and C(13,4):

$$\text{distance} = \sqrt{(13-4)^2 + (4-5)^2} = \sqrt{82} \approx 9.055$$

Therefore, point A is closer to point C.

b) $x^2 + y^2 - 8x + 20y + 107 = 0$
 $\underbrace{x^2 - 8x} + \underbrace{y^2 + 20y} = -107$

Complete the square

$$(x^2 - 8x + 16) + (y^2 + 20y + 100) = -107 + 16 + 100$$

$$(x-4)^2 + (y+10)^2 = 9$$

equation of circle: $(x-h)^2 + (y-k)^2 = r^2$

so the centre (h,k) = (4, -10)

$$r = \text{Radius} = \sqrt{9} = 3$$

10) a) $f(x) = \frac{3}{x^2+9}$

domain: all real numbers

Range: all real numbers

b) $g(x) = \sqrt{4-3x}$

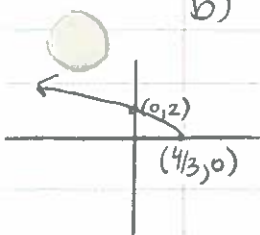
$$4 - 3x \geq 0$$

$$4 \geq 3x$$

$$\frac{4}{3} \geq x$$

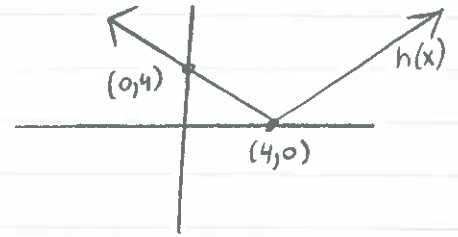
domain: $\{x \mid x \leq 4/3\}$ OR $(-\infty, 4/3]$

Range: $\{y \mid y \geq 0\}$ OR $[0, \infty)$

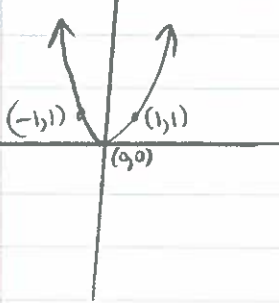


c) $h(x) = |x-4|$

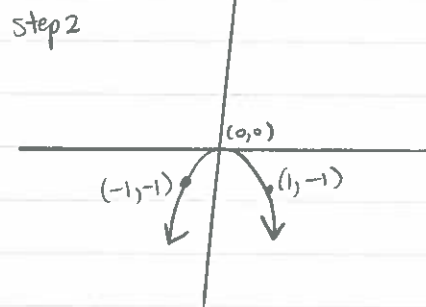
domain: $\{x\}$ all real numbers, $(-\infty, \infty)$
 Range: $\{y \mid y \geq 0\}$ OR $[0, \infty)$



step 1 $g(x) = x^2$

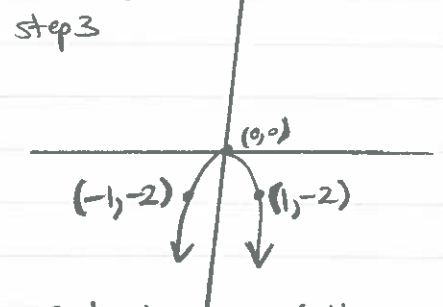


step 2 $g(x) = -x^2$



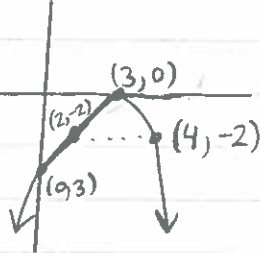
reflect about X-axis

step 3 $g(x) = -2x^2$



note: because of the 2, the parabola is more narrow than in step 2

step 4 $g(x) = -2(x-3)^2$



note the parabola should be same size between steps 3 and 4 (I didn't draw with a scale - you should!)

shift 3 units to the right

12) $f(x) = \frac{x+3}{2x-4}$ $g(x) = \frac{3x-1}{x-2}$

a) $fg = f(x)g(x) = \frac{(x+3)}{2x-4} \cdot \frac{3x-1}{(x-2)} = \frac{(x+3)(3x-1)}{2(x-2)^2}$

b) $\frac{f}{g} = \frac{f(x)}{g(x)} = \frac{x+3}{\frac{2x-4}{\frac{3x-1}{x-2}}}$

domain: $\{x \mid x \neq 2\}$

$\frac{f}{g} = \frac{(x+3)}{2(x-2)} \cdot \frac{(x-2)}{(3x-1)} = \frac{x+3}{2(3x-1)}$

domain: $\{x \mid x \neq 2, x \neq \frac{1}{3}\}$

c) $f \circ g = f(g(x))$

$$f\left(\frac{3x-1}{x-2}\right)$$

$$= \frac{\frac{3x-1}{x-2} + 3}{2\left(\frac{3x-1}{x-2}\right) - 4} = \frac{\frac{3x-1 + 3(x-2)}{x-2}}{\frac{2(3x-1) - 4(x-2)}{x-2}}$$

$$= \frac{6x-7}{x-2} \div \frac{2x+6}{x-2}$$

$$= \frac{6x-7}{x-2} \cdot \frac{x-2}{2(x+3)} = \frac{6x-7}{2(x+3)}$$

domain of $f \circ g$: $\{x \mid x \neq -3, x \neq 2\}$

d) $g \circ f = g(f(x))$

$$= g\left(\frac{x+3}{2x-4}\right) = \frac{3\left(\frac{x+3}{2x-4}\right) - 1}{\left(\frac{x+3}{2x-4}\right) - 2} = \frac{\frac{3(x+3) - 1(2x-4)}{2x-4}}{\frac{x+3 - 2(2x-4)}{2x-4}}$$

$$= \frac{3x+9-2x+4}{2(x-2)} \div \frac{x+3-4x+8}{2x-4}$$

$$= \frac{x+13}{2(x-2)} \cdot \frac{2(x-2)}{-3x+11} = \frac{x+13}{-3x+11}$$

domain: $\{x \mid x \neq 11/3, x \neq 2\}$

13 a) $x = \frac{2y+5}{y+7}$

$$xy + 7x = 2y + 5$$

$$xy - 2y = 5 - 7x$$

$$y(x-2) = 5 - 7x$$

$$f^{-1}(x) = y = \frac{5-7x}{x-2}$$

make sure to
verify that:

① $f^{-1}(f(x)) = x$

② $f(f^{-1}(x)) = x$

K Kepler

b) $f(x)$: vertical asymptotes: $x=2$
 $f^{-1}(x)$: vertical asymptotes: $x=2$

finding oblique/horizontal asymptotes:

$$\begin{array}{r} \underline{f(x)} \\ x+7 \overline{) 2x+5} \\ \underline{-(2x+14)} \\ -9 \end{array}$$

improper rational $f(x)$
 so we need long
 division

$$\text{so } \frac{2x+5}{x+7} = 2 + \frac{-9}{x+7}$$

if $x \rightarrow \infty$, then $y=2$ is a horizontal asymptote
 for $f(x)$

$f^{-1}(x)$

$$\begin{array}{r} x-2 \overline{) -7x+5} \\ \underline{-(-7x+14)} \\ -9 \end{array}$$

$$\frac{5-7x}{x-2} = -7 + \frac{-9}{x-2}$$

if $x \rightarrow \infty$, then $y=-7$ is a horizontal asymptote
 for $f^{-1}(x)$

(14)

	solo	$\frac{1}{t}$
Connie	2 hours	$\frac{1}{2}$
Alvaro	x	$\frac{1}{x}$
Connie + Alvaro	1.25 hours	$\frac{1}{1.25}$

x is the time it takes Alvaro
 to clean the house by himself

$$\text{then, } \frac{1}{2} + \frac{1}{x} = \frac{1}{1.25}$$

1 hour 15 minutes

1 hour + $\frac{15}{60}$

1.25 hours total

$$\frac{1}{x} = 0.3$$

solution:

$$x = 3.33 \text{ hours} \rightarrow 200 \text{ minutes}$$

- ⑮ let X be the amount invested in stocks (\$)
let y be the amount invested in bonds (\$)

$$X + y = 18000$$

$$y = \frac{1}{2}X$$

$$\text{So, } X + \frac{1}{2}X = 18000$$

$$1.5X = 18000$$

$$X = 12000$$

$$y = \frac{1}{2}X = \frac{1}{2}(12000) = 6000$$

Therefore, \$12000 is invested in stocks
\$6000 is invested in bonds

⑯ $P(t) = \frac{1000}{1 + 32.33 e^{-0.439t}}$ t : hours
 $P(t)$: population (grams)

a) $t = 9$

$$P(t) = ? \quad P(t) = \frac{1000}{1 + 32.33 e^{-0.439(9)}} = \underline{\underline{616.57 \text{ grams}}}$$

b) $700 = \frac{1000}{1 + 32.33 e^{-0.439t}}$

$$700(1 + 32.33 e^{-0.439t}) = 1000$$

$$700 + 22631 e^{-0.439t} = 1000$$

$$22631 e^{-0.439t} = 300$$

$$e^{-0.439t} = \frac{300}{22631}$$

$$\ln e^{-0.439t} = \ln \left(\frac{300}{22631} \right)$$

$$-0.439t (\ln e) = \ln \left(\frac{300}{22631} \right)$$

$$t = 9.85 \text{ hours}$$

fun fact:
 $\ln e = 1$

$$c) P(t) = \frac{1000}{1 + 32.33 e^{-0.439t}}$$

The carrying capacity = 1000 grams

carrying capacity: the maximum population size of the species that the environment can sustain indefinitely given the food, habitat, and other resources available in the environment.

→ so as $t \rightarrow \infty$, $e^{-t} = e^{-\infty} = 0$, and $P(t) = \frac{1000}{1+0} = 1000$

So the carrying capacity is 1000 grams

→ So, when $P(t) = \frac{1}{2}(1000) = 500$ what is t ?

$$500 = \frac{1000}{1 + 32.33 e^{-0.439t}}$$

$$(1 + 32.33 e^{-0.439t}) = 2$$

$$32.33 e^{-0.439t} = 1$$

$$e^{-0.439t} = \frac{1}{32.33}$$

fun fact
 $\ln e = 1$

$$\ln e^{-0.439t} = \ln \left(\frac{1}{32.33} \right)$$

$$-0.439t (\ln e) = \ln \left(\frac{1}{32.33} \right)$$

$$-0.439t = \ln \left(\frac{1}{32.33} \right)$$

$$t = \frac{-3.47599}{-0.439} = \underline{\underline{7.92 \text{ hours}}}$$