FAMILY NAME (Capitals) Answer Sheet

First Name (to one version of test)

Signature

Student Number

Mat 1348 Midterm Test

Professor: P. J. Scott Date: Feb. 11, 2010 Time: 80 minutes

Instructions:

(a) *No notes or papers allowed.*

(b) Answer all questions on the test itself (you may write on the backs of pages). There are extra work pages. Write neatly and clearly show your answer.

(c) **Note:** In multiple choice questions, **circle your chosen answer**

(d) Be precise and explain what you are doing in long-answer questions.

(e) The test consists of 10 questions (5 multiple choice, 5 long answer). You may write on the backs of pages. Multiple Choice are worth 2 points each. The others have the indicated marks. Total: 30 points.

Reserved for the Professor

- **Total (30 pts)**

/2	/2	/2	/2	/2
1	2	3	4	5
/3	/4	/5	/4	/4
6	7	8	9	10

1. Consider the set $A = \{1,\ \{1\},\ \{1,2\},\ 2\}$. Among the following sentences, only one is false. **Circle the one false answer.**

   (a) $\emptyset \subseteq A$

   (b) $\{1,\ \{1\}\} \subseteq A$

   (c) $\{2,\ \{1,\ 2\}\} \subseteq A$

   (d) $\{1, 2\} \in A$

   (e) The cardinality of the power set of A is 16.

   [✓ (f) $\{\{1\},\ \{2\}\} \subseteq A$] Since $\{2\} \notin A$

2. Consider the following atoms:

   $P :=$ The program outputs a value.
   $B :=$ The program enters an infinite loop.
   $S :=$ The output value is positive
   $V :=$ The program is infected with a virus.

   Which of the following formulas is a translation of the following:

   > *In order that the output value is positive and that the program is not infected with a virus, it is sufficient that the program does not output a value unless it enters an infinite loop.*

   (a) $((S \wedge \neg V) \rightarrow (\neg P \vee B))$

   (b) $((\neg P \wedge B) \rightarrow (S \vee V))$

   [✓ (c) $((\neg P \vee B) \rightarrow (S \wedge \neg V))$]

   (d) $((S \vee V) \rightarrow (\neg P \wedge B))$

   (e) $(\neg(S \vee V) \rightarrow (\neg P \vee B))$

   (f) None of the above answers.

3. The truth table of a propositional formula φ (whose atoms are A, B and C) is given by:

A	B	C	φ	
T	T	T	T	
T	T	F	T	
T	F	T	T	
T	F	F	F	← $(\neg A \lor B \lor C)$
F	T	T	F	← $(A \lor \neg B \lor \neg C)$
F	T	F	T	
F	F	T	F	← $(A \lor B \lor \neg C)$
F	F	F	T	

Which of the following formulas is a CNF equivalent to φ ?

(a) $((A \land B \land \neg C) \lor (A \land B \land \neg C) \lor (\neg A \land B \land \neg C))$

(b) $((\neg A \land B \land \neg C) \lor (\neg A \land \neg B \land \neg C) \lor (A \land B \land C))$

[✓ (c) $((\neg A \lor B \lor C) \land (A \lor \neg B \lor \neg C) \land (A \lor B \lor \neg C))$]

(d) $((\neg A \land B \land C) \lor (A \land \neg B \land \neg C) \lor (A \land B \land \neg C))$

(e) $((A \lor \neg B \lor \neg C) \land (\neg A \lor B \lor \neg C) \land (A \lor \neg B \lor \neg C))$

(f) $((A \lor B \lor C) \land (\neg A \lor B \lor C) \land (\neg A \lor \neg B \lor C))$

4. Which of the following statements are **true** ?

(1). If X is true, Y is false and Z is false, then $(X \vee (Y \wedge Z))$ is true. ✓

(2). The two formulas $(p \rightarrow q)$ and $(\neg p \rightarrow \neg q)$ are logically equivalent. ✗

(3). The two formulas $(p \wedge (q \vee r))$ and $((p \wedge q) \vee (p \wedge r))$ are logically equivalent. ✓

(4). The formula $(A \rightarrow (\neg A \vee B))$ is a tautology. ✗

Circle the statement which is true

(a) (3) and (4)

(b) (1) only

(c) (4) only

[✓ (d) (1) and (3)]

(e) (2) and (3)

(f) (3) only

5. Let A, B be two finite sets, $f : A \rightarrow B$ and $g : B \rightarrow A$ two functions. Among the following statements, only one is **false. Circle the false statement.**

(a) If $g \circ f$ is injective, then f is injective true

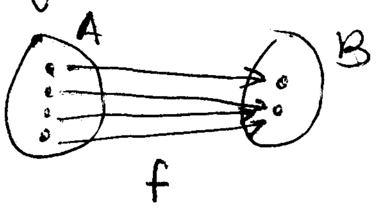
(b) If $g \circ f$ is surjective, then g is surjective true

[✓ (c) If $|A| > |B|$, then f cannot be surjective (recall, $|X|$ is the cardinality of X).]

(d) If f is bijective then $|A| = |B|$ true

(e) If $|A| > |B|$, then f cannot be injective true

Clearly false: Map all of A's elements to B's elements. e.g.



Here $|A| = 4 > |B| = 2$.

6. Let A, B, C be three subsets of a universal set $\mathcal{U}$. Using Boolean algebra operations, prove

$$C \setminus (\overline{A} \cap B) = (C \cap A) \cup (C \setminus B)$$

Recall: $C \setminus X \underset{\text{def}^{\text{n}}}{=} C \cap \overline{X}$

$$C \setminus (\bar{A} \cap B) \underset{\text{def}}{=} C \cap \overline{(\bar{A} \cap B)}$$

$= C \cap (\bar{\bar{A}} \cup \bar{B})$ De Morgan

$= C \cap (A \cup \bar{B})$ since $\bar{\bar{A}} = A$

$= (C \cap A) \cup (C \cap \bar{B})$ Distributivity

$= (C \cap A) \cup (C \setminus B)$ Def$^{\text{n}}$.

7. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by: $f(x, y) = (2x - y, x)$. Prove f is bijective.

Injective: Suppose $f(x,y) = f(x',y')$. ⊛
We must prove $x = x'$ and $y = y'$.
From ⊛, $(2x-y, x) = (2x'-y', x')$. So

$$2x - y = 2x' - y' \quad ①$$
$$x = x' \quad ②$$

Plugging ② into ①, $2x - y = 2x - y' \Rightarrow y = y'$. ③
Hence we get (from ② and ③) $x = x'$ and $y = y'$.

Surjective: We must show:
for all $(u,v) \in \mathbb{R} \times \mathbb{R}$, There exists $(x,y) \in \mathbb{R} \times \mathbb{R}$ satisfying $f(x,y) = (u,v)$. So we suppose we have the equation $f(x,y) = (u,v)$.

Then
$$\left.\begin{aligned} 2x - y &= u \\ x &= v \end{aligned}\right\} \Rightarrow \begin{aligned} -y &= u - 2x = u - 2v \\ \therefore y &= 2v - u \end{aligned}$$

Hence we get $\boxed{x = v}$ $\boxed{y = 2v - u}$

Check: $f(x,y) = f(v, 2v - u)$
$$= (2v - (2v - u), v)$$
$$= (u, v) \quad ✓$$

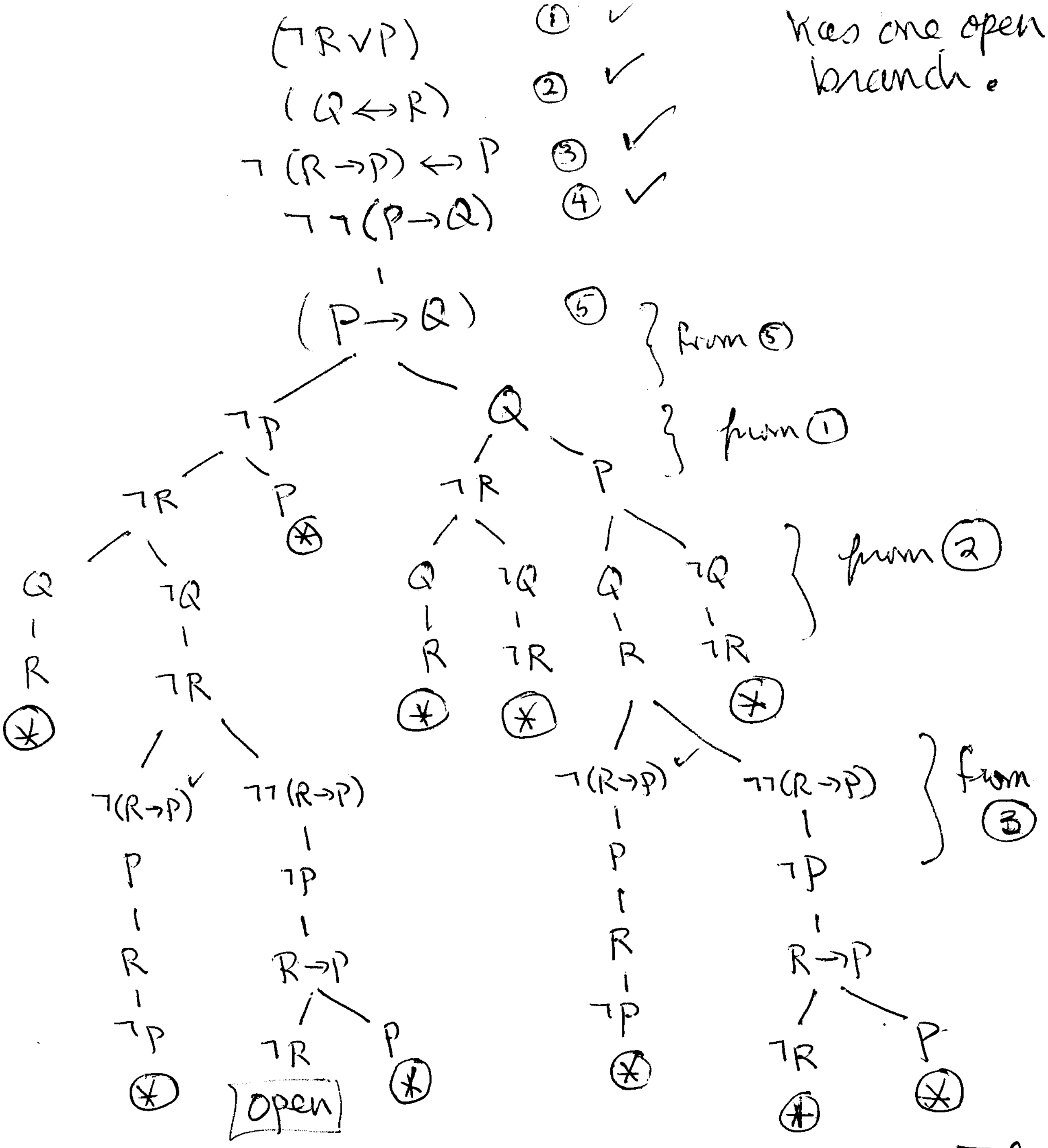
Hence f is surjective. $\therefore$ f is bijective.

8. Consider the following argument. Using any method we gave in class, verify if the argument is valid or not. If it is not valid, give all values of the atoms which falsify the argument.

$$\begin{array}{c}(\neg R \vee P)\\(Q \leftrightarrow R)\\(\neg(R \rightarrow P) \leftrightarrow P)\\\hline \neg(P \rightarrow Q)\end{array}$$

} The argument is not valid since the tableau has one open branch.

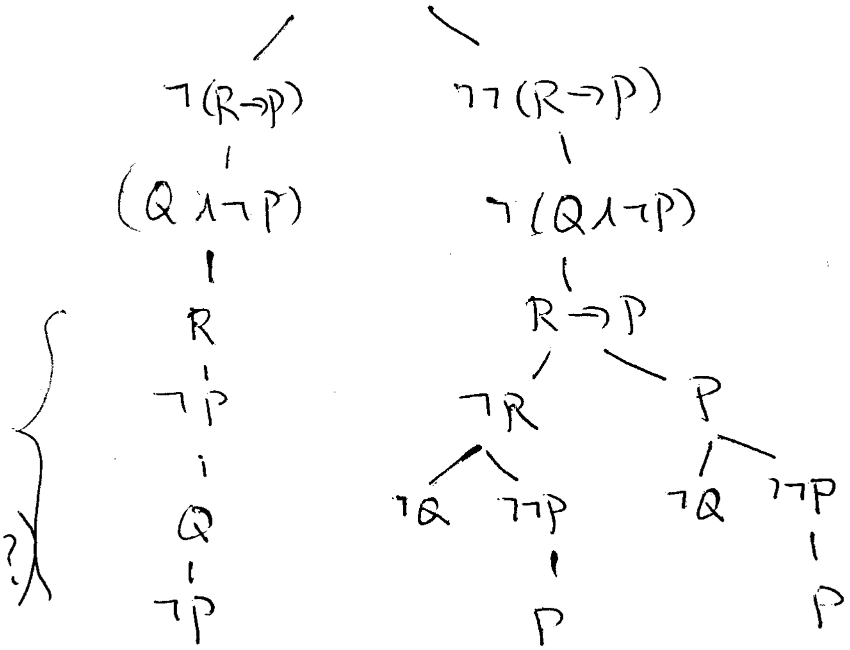
Let's use tableaux:



∴ Valuation: P=Q=R=false makes argument False.

9. Use whatever methods you prefer from class to find a DNF equivalent to the following formula:

$$(\neg(R \to P) \leftrightarrow (Q \wedge \neg P)).$$



We can ignore duplicate letters (why?)

$$\equiv \boxed{(\neg P \wedge Q \wedge R) \vee (\neg Q \wedge \neg R) \vee (P \wedge \neg R) \vee (\neg Q \wedge P) \vee P}$$

the Boolean algebra method and the truth table method are also possible, but much longer.

10. You are on the island of knights and knaves. You encounter two individuals A and B who say the following:

A: *If there is not a hotel on the island then B is a knave.*

B: *There is a hotel on the island and at least one of us is a knave.*

Questions:

(i) Is it possible to determine the nature of A and B (knights or knaves)?

(ii) Is it possible to determine if there is a hotel on the island?

Case 1. A Knave, Then A speaks F. Therefore there's not a hotel and B is Knight. B says "at least one of us is a Knave": That's T. But also "there is a hotel" must be T (since B speaks T). Contradiction.

Hence A must be a Knight.

Case 2: A Knight. So A speaks T. So "there is not hotel implies B knave" is T. From Tableau rules we see 2 subcases:

Subcase 2.1. There is a hotel. Notice B cannot be a Knight (since he says at least one is a Knave & he speaks T). Hence B is Knave. But B speaks True! Contradiction.

Subcase 2.2 B Knave. But "at least one of us is knave" is T. ∴ cannot be Hotel

CONSISTENT: $\boxed{\text{A Knight, B Knave, No Hotel.}}$

9. You are on the island of knights and knaves. You encounter two individuals A and B who say the following:

   A: *If there is not a hotel on the island then B is a knave.*

   B: *There is a hotel on the island and at least one of us is a knave.*

   Questions:

   (i) Is it possible to determine the nature of A and B (knights or knaves)?

   (ii) Is it possible to determine if there is a hotel on the island?

More Symbolic Analysis : { Hotel := "there is hotel"
B Knave := B is a Knave
B knight := B is a knight

A : $(\neg$ Hotel $\rightarrow$ (B Knave))

B : Hotel $\wedge$ (at least one Knave)

Case 1 : A Knave. ∴ $(\neg$ Hotel $\rightarrow$ B Knave$)=$ F
∴ $\neg$ Hotel $=$ T, B Knave $=$ F ∴ B Knight.
∴ "at least one Knave" $=$ T and (from B)
we have Hotel $=$ T. But $\neg$ Hotel $=$ T. contradiction

Case 2 : A Knight. ∴ $(\neg$ Hotel $\rightarrow$ B Knave$)=$ T.
2.1 : $\neg\neg$ Hotel i.e. Hotel is T. If B Knight, then B speaks T $\wedge$ T $=$ T. But impossible, since he says B knave. Hence B Knave. Also impossible, since B speaks T. ∴ This subcase impossible.
2.2 B Knave ∴ B speaks F. ∴ $\neg$ Hotel $=$ T. (since "at least one knave" is T). CONSISTENT.

$\boxed{\text{∴ A KNIGHT, B KNAVE, NO Hotel}}$