MATH 3705* Test 4 - SOLUTIONS March 2015

1. [6 marks] Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2,\ 0 < y < 1$, which satisfies the boundary conditions

$$u(0,y) = 2y, \quad u(2,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = 2 - x.$$

Write down the complete solution $u(x,y)$.

Solution:

The boundary condition is continuous, and linear on each portion of the boundary, so

$$u(x,y) = \alpha x + \beta y + \gamma xy + \delta.$$

(1) The lower horizontal segment: $y = 0 \ \Rightarrow\ \alpha x + \delta = 0 \ \Rightarrow\ \alpha = \delta = 0$
$\Rightarrow\ u(x,y) = \beta y + \gamma xy.$

(2) The left vertical segment: $x = 0 \ \Rightarrow\ \beta y = 2y \ \Rightarrow\ \beta = 2$
$\Rightarrow\ u(x,y) = 2y + \gamma xy.$

(3) The right vertical segment: $x = 2 \ \Rightarrow\ 2y + 2\gamma y = 0 \ \Rightarrow\ \gamma = -1$
$\Rightarrow\ u(x,y) = 2y - xy.$

Check the upper horizontal segment: $y = 1 \ \Rightarrow\ 2 - x = 2 - x$. Thus,

$$u(x,y) = 2y - xy.$$

2. [6 marks] The solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ inside the circle $r = 3$ has the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

Find the solution of Laplace's equation inside the circle $r = 3$, subject to the boundary condition $u(3,\theta) = 1 + 2\sin(3\theta) - 3\cos(2\theta)$.

Solution:

$$1 + 2\sin(3\theta) - 3\cos(2\theta) = u(3,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 3^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

$\Rightarrow \frac{a_0}{2} = 1,\ 3^2\, a_2 = -3,\ 3^3\, b_3 = 2$, so $a_2 = -\frac{1}{3}$, $b_3 = \frac{2}{27}$, and $a_n = b_n = 0$ otherwise. Thus,

$$u(r,\theta) = 1 - \frac{1}{3}r^2\cos(2\theta) + \frac{2}{27}r^3\sin(3\theta)$$

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3. [4 marks] The solution of the wave equation $u_{xx} = \frac{1}{9}u_{tt}$, $0 < x < 2$, which satisfies the boundary conditions $u(0,t) = u(2,t) = 0$, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left\{a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right)\right\}.$$

If $u(x,t)$ satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 3\sin(\pi x) - \sin(3\pi x)$, find the coefficients a_n and b_n. Write down the complete solution.

Solution:

$b_2 = \frac{1}{\pi}$, $b_6 = -\frac{1}{9\pi}$, $b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

$$u(x,t) = \frac{1}{\pi}\sin(\pi x)\sin(3\pi t) - \frac{1}{9\pi}\sin(3\pi x)\sin(9\pi t).$$

4. [14 marks] The solution of the heat equation $w_{xx} = \frac{1}{\alpha^2} w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \frac{1}{9} u_t,\ 0 < x < 2$, which satisfies the boundary conditions $u(0,t) = 2,\ u(2,t) = 4$, and the initial condition $u(x,0) = 5$. Write down the complete solution $u(x,t)$ (give the first four terms).

Solution:

$L = 2,\ \alpha = 3$. Let $u(x,t) = v(x) + w(x,t)$, where w satisfies $w_{xx} = \frac{1}{9} w_t,\ 0 < x < 2$, $w(0,t) = w(2,t) = 0$, and $v(x)$ satisfies $v''(x) = 0,\ v(0) = 2,\ v(2) = 4$. Then $v(x) = ax + b,\ v(0) = 2 \ \Rightarrow b = 2 \ \Rightarrow v(x) = ax + 2$. Since $v(2) = 4$ then $2a+2 = 4 \Rightarrow a = 1 \Rightarrow v(x) = x+2$. Next, $w(x,0) = u(x,0) - v(x) = 5 - (x-2) = 3 - x$, so

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{9n^2\pi^2}{4} t},$$

with $3 - x = w(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \ \Rightarrow$

$$b_n = \frac{2}{2} \int_0^2 (3-x) \sin\left(\frac{n\pi x}{2}\right) dx = \left\{-\frac{2}{n\pi}(3-x)\cos\left(\frac{n\pi x}{2}\right)\right\}_0^2 - \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx =$$

$= \frac{2}{n\pi}\left[3 - (-1)^n\right], n \geq 1$. Thus,

$$u(x,t) = x + 2 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[3 - (-1)^n\right] \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{9n^2\pi^2}{4} t}$$

$$= x + 2 + \frac{8}{\pi} \sin\left(\frac{\pi x}{2}\right) e^{-\frac{9\pi^2}{4} t} + \frac{2}{\pi} \sin(\pi x) e^{-9\pi^2 t} + \ldots$$

Marking guidelines:

Problem 1. 1 mark for the general form of the solution; 1 marks for each correct coefficient; 1 mark for the complete solution.

Problem 2. 2 marks for each correct coefficient (a_1 and b_2), 2 marks for the final solution.

Problem 3. 2 marks for each correct coefficient.

Problem 4:

1 mark for identifying L and α,
1 mark for $u(x,t) = v(x) + w(x,t)$,
3 marks for finding $v(x)$,
1 marks for the correct BCs for $w(x,t)$,
5 marks for computing b_n,
2 marks for the final solution,
1 mark for the first three terms.