MATH 3705* B Test 3 Answers and solutions March 2015

Questions 1-5 are multiple choice.

1. [3] The general solution of $xy'' + y' + 7xy = 0$ for $x > 0$ is

   (a) $c_1 J_0(\sqrt{7}\,x) + c_2 J_0(\sqrt{7}\,x)\ln(x)$ (b) $c_1 J_0(\sqrt{7}\,x) + c_2 Y_0(\sqrt{7}\,x)$

   (c) $c_1 J_{\sqrt{7}}(x) + c_2 J_{-\sqrt{7}}(x)$ (d) $c_1 J_{\sqrt{7}}(x) + c_2 Y_{\sqrt{7}}(x)$ (e) None of these

2. [3] The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ for $x > 0$ is

   (a) $c_1 J_3(\sqrt{5}\,x) + c_2 J_{-3}(\sqrt{5}\,x)$ (b) $c_1 J_3(\sqrt{5}\,x) + c_2 Y_3(\sqrt{5}\,x)$

   (c) $c_1 J_{\sqrt{5}}(3x) + c_2 J_{-\sqrt{5}}(3x)$ (d) $c_1 J_{\sqrt{5}}(3x) + c_2 Y_{\sqrt{5}}(3x)$ (e) None of these

3. [2] At $x = 20$, the Fourier Cosine series of the 8-periodic function $f(x) = \begin{cases} 3, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$ converges to

   (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) 3 (e) None of these

4. [2] The function $x^3\sin(x) + x^2\cos(x)$ is

   (a) even; (b) odd; (c) neither .

5. [2] If f is a periodic function of period 3, and $f(x) = 2x^2,\ 3 > x \geq 0$, then $f(97)$ is

   (a) 18; (b) 8; (c) -8; (d) 2; (e) 0.

Answers: b, b, a, a, d.

6. [18 marks] Find the full Fourier series of the 2π-periodic function $f(x) = \begin{cases} x, & 0 < x \le \pi \\ 1, & \pi < x \le 2\pi \end{cases}$

Give the first three terms of the series explicitly, i.e, the terms $\frac{a_0}{2}$, $a_1 \cos\left(\frac{n\pi x}{L}\right)$ and $b_1 \sin\left(\frac{n\pi x}{L}\right)$ with the appropriate a_0, a_1, b_1, n and L.

Solution:

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} 1\,dx\right\} = \frac{\pi}{2} + 1 = \frac{\pi+2}{2}.$$

For $n \ge 1$,

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\cos(nx)\,dx + \int_{\pi}^{2\pi} \cos(nx)\,dx\right\} = \frac{1}{\pi}\left\{\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right\}_0^{\pi} + \frac{1}{\pi}\left\{\frac{1}{n}\sin(nx)\right\}_{\pi}^{2\pi} =$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right] + 0 = \frac{1}{n^2\pi}[(-1)^n - 1].$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(nx)\,dx + \int_{\pi}^{2\pi} \sin(nx)\,dx\right\} = \frac{1}{\pi}\left\{-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right\}_0^{\pi} + \frac{1}{\pi}\left\{-\frac{1}{n}\cos(nx)\right\}_{\pi}^{2\pi} =$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(n\pi)\right] - \frac{1}{n\pi}\left[\cos(2n\pi) - \cos(n\pi)\right] = \frac{1}{n\pi}\left[-\pi(-1)^n - 1 + (-1)^n\right].$$

To give the first three terms, we will need $\frac{a_0}{2} = \frac{\pi+2}{4}$, $a_1 = -\frac{2}{\pi}$, $b_1 = \frac{\pi-2}{\pi}$.

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The full Fourier series is given by

$$\frac{\pi+2}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi}\Big((-1)^n - 1\Big)\cos(nx) + \frac{1}{n\pi}\Big(-\pi(-1)^n - 1 + (-1)^n\Big)\sin(nx) =$$

$$\frac{\pi+2}{4} - \frac{2}{\pi}\cos(x) + \frac{\pi-2}{\pi}\sin(x) + \ldots$$