MATH 3705* B Test 1 January 2015

LAST NAME: ________________________________ **ID#:**________________

Questions 1-5 are multiple choice. Circle the correct answer. Only the answer will be marked. [**12 marks**].

1. [**2**] $\mathcal{L}\{e^{-2t}\cos(3t)\} =$

$(a)\ \dfrac{s+2}{(s+2)^2+9}$ $(b)\ \dfrac{s}{s^2+9}$ $(c)\ \dfrac{s-2}{(s-2)^2+9}$ (d) None of the above

2. [**2**] $\mathcal{L}\{t\sin(3t)\} =$

$(a)\ \dfrac{6s}{(s^2+9)^2}$ $(b)\ \dfrac{s^2-9}{(s^2+9)^2}$ $(c)\ \dfrac{9-s^2}{(s^2+9)^2}$ (d) None of the above

3. [**2**] $\mathcal{L}\{u(t-1)\,e^{2t}\} =$

$(a)\ \dfrac{e^{-s}}{s-2}$ $(b)\ \dfrac{e^{1-s}}{s+2}$ $(c)\ \dfrac{e^{2-s}}{s-2}$ (d) None of the above

4. [**3**] $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2-25}\right\} =$

$(a)\ e^{5t}-e^{-5t}$ $(b)\ e^{5t}+e^{-5t}$ $(c)\ -\cos(5t)$ (d) None of the above

5. [**3**] $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4}\right\} =$

$(a)\ u(t-3)\,\cos(2t)$ $(b)\ u(t+3)\,\cos[2(t+3)]$

$(c)\ u(t-3)\,\cos[2(t-3)]$ (d) None of the above

Answers: 1.(a), 2.(a), 3.(c), 4.(b), 5.(c).

6. [**5 marks**] Let $f(t) = e^t$ for $0 < t < 1$ and $f(t+1) = f(t)$ for all $t \geq 0$. Find $\mathcal{L}\{f(t)\}$.

Solution:

Since f is periodic with the period $\omega = 1$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 e^t\, e^{-st}\, dt = \frac{1}{1-e^{-s}} \int_0^1 e^{t(1-s)}\, dt = \frac{1}{1-e^{-s}} \left\{ \frac{1}{1-s}\, e^{t(1-s)} \right\}_0^1 =$$

$$= \frac{1}{1-e^{-s}} \left(\frac{1}{1-s} \right) (e^{1-s} - e^0) = \frac{e^{1-s} - 1}{(1-e^{-s})(1-s)}.$$

7. [**6 marks**] Employ the Laplace transform to solve the initial-value problem

$$y'' + 4y' + 8y = 0, \quad y(0) = 2,\ y'(0) = 4.$$

Solution:

$$[s^2 Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 8Y(s) = 0$$

$$\Rightarrow (s^2 + 4s + 8)Y(s) - 2s - 12 = 0 \Rightarrow Y(s) = 2\frac{s+6}{s^2+4s+8} = 2\frac{(s+2)+4}{(s+2)^2+4} =$$

$$2\frac{s+2}{(s+2)^2+4} + 4\frac{2}{(s+2)^2+4} \quad \Rightarrow \quad y(t) = 2e^{-2t}\cos(2t) + 4e^{-2t}\sin(2t).$$

8. [**7 marks**] Find $f(t)$, if

$$f(t) - 2\int_0^t f(x)\,dx = 8t.$$

Solution:

Let $\mathcal{L}\{f(t)\} = F(s)$. Take the Laplace transform of both parts of the equation:

$$F(s) - 2\frac{1}{s}F(s) = 8\frac{1}{s^2} \quad \Rightarrow F(s)\left(1 - \frac{2}{s}\right) = \frac{8}{s^2} \quad \Rightarrow F(s) = \frac{8}{s(s-2)}.$$

To invert $F(s)$, one can use **(a)** partial fractions or **(b)** $\mathcal{L}\left\{\int_0^t g(x)\,dx\right\} = \frac{G(s)}{s}$.

(a) $F(s) = \dfrac{8}{s(s-2)} = -\dfrac{4}{s} + \dfrac{4}{s-2} \quad \Rightarrow \quad y(t) = -4 + 4e^{2t}.$

OR

(b) $\dfrac{8}{s(s-2)} = \dfrac{1}{s}\cdot\dfrac{8}{s-2} = \dfrac{1}{s}\cdot G(s).$ Then $\mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{8}{s-2}\right\} = 8e^{2t} = g(t),$

and

$$y(t) = \int_0^t g(x)\,dx = 8\int_0^t e^{2x}\,dx = 8\cdot\frac{1}{2}e^{2x}\Big|_0^t = 4e^{2t} - 4.$$

Table of Laplace Transforms

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,\ s > 0$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1, \text{ and } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),\ s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),\ s > a \geq 0$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0),\ n \geq 0$$

$$\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s) \equiv (-1)^n\frac{d^n}{ds^n}F(s),\ n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t)*g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s), \text{ where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as},\ a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-\omega s}}\int_0^\omega e^{-st}f(t)\,dt \text{ whenever } f \text{ is periodic with period } \omega$$