MAT 1320F Calculus I
Midterm 2

Professor: C. Rada

Family Name: -SOL-

First Name: ______________________

Student Number: ______________________

1	2	3	4	5	6	7	Total

- Time: 80 min.

- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.

- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.

- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

- During any test or exam, you must not have any type of electronic device. Having such a device is considered academic fraud.

0.5p

$$\ln f(x) = \ln (\cos x)^x$$
$$\ln f(x) = x \cdot \ln (\cos x)$$

1. [2 points] Find the derivative of $f(x) = (\cos x)^x$.

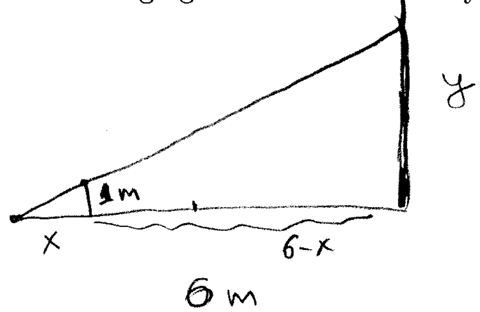
$$\frac{f'(x)}{f(x)} = 1 \cdot \ln(\cos x) + x \cdot \frac{-\sin x}{\cos x}$$

$$f'(x) = (\cos x)^x \cdot \left[1 \cdot \ln(\cos x) + x \frac{-\sin x}{\cos x} \right]$$

0.5p

2. [2 points] A spotlight on the ground shines on a wall 6 m away. A boy of height 1 m is walking from the spotlight toward the wall. How fast the length of his shadow on the wall is changing when he is 4 m away from the light and walking at a speed 0.5 m/s?

1p



$$\frac{1}{y} = \frac{x}{6} \; ; \; \text{so: } y = \frac{6}{x}$$

$$\frac{dy}{dt} = 6(-1)\,x^{-2}\,\frac{dx}{dt}$$

AT 4 m away: $\quad 6(-1)\,2^{-2}\,(0.5)\ \frac{m}{s} = -0.75\ \frac{m}{s}$

$$-\frac{6}{4} \times \frac{1}{2} = -\frac{6}{8} = -\frac{3}{4}$$

1p

$f'(x) = x^2 + x$

$f(x) = \frac{1}{3}x^3 + \frac{x^2}{2} + 3$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -3 - \frac{f(-3)}{f'(-3)} = \boxed{-2.5833}$

0.5p

3. [1 point] Use Newton's method to find x_3 if $x_1 = -3$, and $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3 = 0$. (Give your answer to 4 decimal places!)

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \boxed{-3.2157}$

0.5p

0.5p

4. [5 points]

(a) Find the limit $\lim_{x\to 0+} x^{2x} = L \Rightarrow \ln\left(\lim_{x\to 0^+} x^{2x}\right) = \ln L \Rightarrow$

$\lim_{x\to 0^+} \ln x^{2x} = \ln L \Rightarrow \lim_{x\to 0^+} 2x \cdot \ln x = \ln L \Rightarrow 2 \cdot \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \ln L$

$2 \cdot \lim_{x\to 0^+} \frac{\frac{1}{x}}{(-1)x^{-2}} = \ln L \Rightarrow 2 \cdot \lim_{x\to 0^+} \frac{x}{(-1)} = \ln L$

$0 = \ln L \Rightarrow L = e^0 = \boxed{1}$

0.5p

(b) If $f : [0, 2\pi] \to \mathbf{R}$, is given by $f(x) = \cos^2(x) - 2\sin(x)$, find the intervals on which f is increasing, and intervals on which f is decreasing.

$f'(x) = 2\cos x(-\sin x) - 2\cos x = -2\cos x\left[\cancel{f}\ \sin x + 1\right]$

$f'(x) = 0 \iff x = \frac{\pi}{2}, \frac{3\pi}{2},$

0.5p

x	0		$\frac{\pi}{2}$		$3\pi/2$		2π
$f'(x)$		$\cancel{+}$	0	$+$	0	$\cancel{+}$	
$f(x)$		↗		↘		↗	
		↘		↗		↘	

$\leftarrow$ 0.5p

(c) If $f : [0, 2\pi] \to \mathbf{R}$, is given by $f(x) = \cos^2(x) - 2\sin(x)$, find the local maximums, local minimums (if any).

$\frac{\pi}{2}$ = LOCAL ~~MAX~~ MIN 0.5

$y' = -2\sin x \cos x - 2\cos x < 0$

$2\cos x(\sin x + 1) < 0$

$\frac{3\pi}{2}$ = LOCAL ~~MIN~~ MAX 0.5

$\cos x < 0$

$\sin x < -1$

0.5p

(d) If $f : [0, 2\pi] \to \mathbf{R}$, is given by $f(x) = \cos^2(x) - 2\sin(x)$, find the intervals of concavity, and inflection points (if any).

$1 - \sin^2 x$

$2\sin^2 x - \cos^2 x + 2\sin x$

$f''(x) = 2[-\sin x \cdot (-\sin x) + \cos x \cdot (-\cos x)] - 2 \cdot (-\sin x)$

$f''(x) = 2 + 2\sin x = 2(1 + \sin x)$; $f''(x) = 0 \iff \sin x = -1$ or $x = \frac{3\pi}{2}$

x	0	$3\pi/2$	2π
$f''(x)$	$+ +$	0	$+ +$
$f(x)$			

0.5p

$2(\sin^2(x) + \sin(x) - \cos^2 x)$

$\sin^2(x) + \sin(x) = \cos^2 x$

$1 - \sin^2 x$

$2\sin^2(x) + \sin(x) = 1$

(e) If $f : [-1, 1] \to \mathbf{R}$, is given by $f(x) = \ln(x^2 + x + 1)$ find the absolute maximum and absolute minimum.

$\boxed{1.0986}$

$\frac{-1 \pm \sqrt{3}}{2}$ $2x^2 + x - 1 = 0$ $\Delta = 1 - 4(2)(-1)$ 9

0.25p $f(1) = \ln 3$, $f(-1) = \ln 1 = \boxed{0}$

$f'(x) = \frac{2x + 1}{x^2 + x + 1}$; $f'(x) = 0 \Rightarrow \boxed{x = -\frac{1}{2} \in [-1, 1]}$

0.25p $f(-\frac{1}{2}) = \ln\left(\frac{1}{4} + (-\frac{1}{2}) + 1\right) = \ln\left(\frac{3}{4}\right) = -0.28\cancel{77}$

0.25p ABS. MAX: $(1, \ 1.0986)$

ABS. MIN: $(-\frac{1}{2}, \ -0.2877)$

0.25p

$L(x) = f(0) + f'(0)(x-0) = 3 + \frac{1}{6}x$ 1p

5. [2 points] Find the linear approximation of $f(x) = \sqrt{9+x}$ at $x = 0$ and use it to estimate $\sqrt{9.5}$.

$f(0) = \sqrt{9} = 3$

$f'(x) = \frac{1}{2}(9+x)^{-\frac{1}{2}}$

$f'(0) = \frac{1}{6}$

0.5p

$\sqrt{9.5} = f(0.5) \approx L(0.5) = 3 + \frac{1}{6}(0.5) =$

$= \boxed{3.0833}$ 0.5p

6. [2 points] Use $n = 4$ intervals and the right endpoints as sample points to estimate the area under the curve $y = \dfrac{1}{1+4x}$ between $x = 0$ and $x = 1$.

1p $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ $x_1^* = 0.25$, $x_2^* = 0.5$, $x_3^* = 0.75$, $x_4^* = 1$

$A(R) \approx \frac{1}{4}\left[\frac{1}{1+4\cdot(0.25)} + \frac{1}{1+4\cdot(0.5)} + \frac{1}{1+4\cdot(0.75)} + \frac{1}{1+4\cdot(1)}\right]$

$= 0.320833$ 1p

7. [1 point] Find $f(x)$ if $f'(x) = \cos(x) + \sin(x) + 1$ and $f(0) = 1$.

$f(x) = \sin x - \cos x + x + C$ 0.5p

$1 = f(0) = 0 - 1 + 0 + C \Rightarrow C = \boxed{2}$ 0.5p

Rough WORK: do NOT detach!

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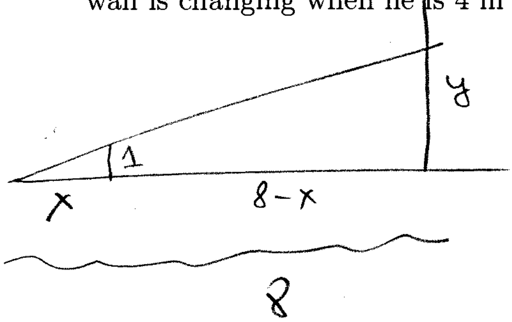
$$\frac{f'(x)}{f(x)} = 1 \cdot \ln(\sin x) + x \cdot \frac{\cos x}{\sin x}$$

$$f'(x) = (\sin x)^x \left[1 \cdot \ln(\sin x) + x \cdot \frac{\cos x}{\sin x} \right]$$

2. [2 points] A spotlight on the ground shines on a wall 8 m away. A boy of height 1 m is walking from the spotlight toward the wall. How fast the length of his shadow on the wall is changing when he is 4 m away from the light and walking at a speed 0.5 m/s?



$\frac{1}{y} = \frac{x}{8}$; So: $y = \frac{8}{x}$

$$\frac{dy}{dt} = 8(-1)x^{-2}\frac{dx}{dt}$$

1p

AT 4m AWAY: $8(-1) \cdot 4^{-2} \cdot (0.5) \frac{m}{s} = -0.25 \frac{m}{s}$

1p

$f'(x) = x^2 + x$; $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \boxed{}$

3. [1 point] Use Newton's method to find x_3 if $x_1 = -3$, and $\frac{1}{3}x^3 + \frac{1}{2}x^2 = -3$. (Give your answer to 4 decimal places!)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \boxed{}$$

4. [5 points]

(a) Find the limit $\lim_{x \to 0+} x^{6x}$ $= L \Rightarrow \ln\left(\lim_{x\to 0^+} x^{6x}\right) = \ln L \Rightarrow$

$$\lim_{x\to 0^+} \ln x^{6x} = \ln L \Rightarrow \lim_{x\to 0^+} 6x \cdot \ln x = \ln L \Rightarrow$$

$$6 \cdot \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \ln L \Rightarrow 6 \lim_{x\to 0^+} \frac{\frac{1}{x}}{(-1)x^{-2}} = \ln L \Rightarrow$$

$$6 \lim_{x\to 0^+} \frac{x}{-1} = \ln L \Rightarrow 0 = \ln L \Rightarrow L = e^0 = \boxed{1}$$

(b) If $f : [0, 2\pi] \to \mathbf{R}$, is given by $f(x) = \cos^2(x) - 2\sin(x)$, find the intervals on which f is increasing, and intervals on which f is decreasing.

(c) If $f : [0, 2\pi] \to \mathbf{R}$, is given by $f(x) = \cos^2(x) - 2\sin(x)$, find the local maximums, local minimums (if any).

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(e) If $f : [-1, 2] \to \mathbf{R}$, is given by $f(x) = \ln(x^2 + x + 1)$ find the absolute maximum and absolute minimum.

$L(x) = f(0) + f'(0)(x-0) = 3 + \frac{1}{6}x$

5. [2 points] Find the linear approximation of $f(x) = \sqrt{9+x}$ at $a = 0$ and use it to estimate $\sqrt{9.3}$.

$f(0) = \sqrt{9} = 3$

$f'(x) = \frac{1}{2}(9+x)^{-\frac{1}{2}}$

$f'(0) = \frac{1}{6}$

$\sqrt{9.3} = f(0.3) \approx L(0.3) = 3 + \frac{1}{6}(0.3) = \boxed{3.05}$

6. [2 points] Use $n = 4$ intervals and the right endpoints as sample points to estimate the area under the curve $y = \dfrac{1}{1+2x}$ between $x = 0$ and $x = 1$.

$\Delta x = \frac{1-0}{4} = \frac{1}{4}$; $x_1^* = 0.25$, $x_2^* = 0.5$, $x_3^* = 0.75$, $x_4^* = 1$

$A(R) \approx \frac{1}{4}\left[\frac{1}{1+2(0.25)} + \frac{1}{1+2(0.5)} + \frac{1}{1+2(0.75)} + \frac{1}{1+2(1)}\right]$

$= 0.475$ $\quad \frac{1}{1.5} \quad \frac{1}{2} \quad \frac{1}{2.5} \quad \frac{1}{3}$

7. [1 point] Find $f(x)$ if $f'(x) = \cos(x) - \sin(x) + 2$ and $f(0) = 2$.

$f(x) = \sin x + \cos x + 2x + C$, $C \in \mathbb{R}$ $\rightarrow 0.5p$

$2 = f(0) = 0 + 1 + 0 + C \Rightarrow C = 1$ $\rightarrow 0.5p$

Rough WORK: do NOT detach!