

ENGR 213, Section XX, Fall Semester 2014, Midterm Test 2, Solutions

**Problem 1.** The differential equation

$$(1 + x^2)y'' - 2xy' + 2y = 0$$

has a solution  $y_1(x) = x$ . Find another solution  $y_2(x)$  such that the solution set  $\{y_1(x), y_2(x)\}$  is a fundamental set of solutions for the given equation on  $(-\infty, \infty)$ .

**Solution 1 by using the ready formula in the Method of Reduction of the Order.** First, standard form to determine  $p(x)$ :

$$y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0 \Rightarrow \mathbf{p(x) = -\frac{2x}{1+x^2}}.$$

Second, by using the ready formula with  $p(x) = -\frac{2x}{1+x^2}$  and  $y_1(x) = x$  we obtain

$$\begin{aligned} \mathbf{y_2} &= \mathbf{y_1} \int \frac{e^{-\int \mathbf{p(x)}dx}}{\mathbf{y_1^2(x)}} \mathbf{dx} = x \int \frac{e^{\frac{2x}{1+x^2}dx}}{x^2} dx = x \int \frac{e^{\ln(1+x^2)}}{x^2} dx \\ &= x \int \frac{1+x^2}{x^2} dx = x \int \left( \frac{1}{x^2} + 1 \right) dx = x \left( -\frac{1}{x} + x \right) = x^2 - 1 \end{aligned}$$

Hence,

$$\mathbf{y_1(x) = x, \quad y_2(x) = x^2 - 1.}$$

Now, taking the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix} = 2x^2 - (x^2 - 1) = x^2 + 1 \neq 0 (> 0)$$

for all  $x \in (-\infty, \infty)$ . Hence, **the solution set  $y_1$  and  $y_2$  form a fundamental set of solutions for the given differential equation in  $(-\infty, \infty)$ .**

**Another way to show that  $y_1$  and  $y_2$  form a fundamental set:** Suppose now that  $y_1$  and  $y_2$  are linearly dependent on  $(-\infty, \infty)$ . Then, there are  $c_1$  and  $c_2$  such that at least one of them is nonzero and  $c_1x + c_2(1 - x^2) = 0$ ,  $x \in (-\infty, \infty)$  and this is possible only when  $c_1 = c_2 = 0$ . Hence,  $y_1$  and  $y_2$  are linearly independent on the interval  $(-\infty, \infty)$  and from here they form a fundamental set of solutions for the given differential equation on  $(-\infty, \infty)$ .

**Solution 2 following the theory.** We are looking for a second solution in the form:

$$y_2(x) = y_1(x)u(x), \quad y_1 = x.$$

Represent the differential equation in standard form:

$$y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0.$$

Plug in the DE  $y_2(x) = y_1(x)u(x)$ :

$$[y_1''u + 2y_1'u' + y_1u''] - \frac{2x}{1+x^2}[y_1u' + y_1'u] + \frac{2}{1+x^2}y_1u = 0$$

and use the fact that  $y_1$  is a solution to simplify the above differential equation:

$$2y_1'u' + y_1u'' - \frac{2x}{1+x^2}y_1u' = 0.$$

Using the explicit form of  $y_1(x) = x, y_1'(x) = 1$  we obtain a DE for  $u$ :

$$\begin{aligned} 2u' + xu'' - \frac{2x^2}{1+x^2}u' &= 0 \quad \Rightarrow \quad xu'' + \frac{2}{1+x^2}u' = 0 \\ \Rightarrow \quad u' = v, \quad xv' + \frac{2}{1+x^2}v &= 0 \quad \Rightarrow \quad \frac{dv}{v} = -\frac{2}{x(1+x^2)}dx \\ \Rightarrow \quad \text{partial fractions } \int \frac{dv}{v} &= \int \left( -\frac{2}{x} + \frac{2x}{1+x^2} \right) dx \\ \Rightarrow \quad \ln(v) = -2\ln(x) + \ln(1+x^2) &\Rightarrow \quad v = \frac{1+x^2}{x^2} = \frac{1}{x^2} + 1 \\ \Rightarrow \quad u' = \frac{1}{x^2} + 1 \quad \Rightarrow \quad u = \int \left( \frac{1}{x^2} + 1 \right) dx &\Rightarrow \quad u = -\frac{1}{x} + x \\ \Rightarrow \quad y_2 = y_1u = x \left( -\frac{1}{x} + x \right) &= x^2 - 1. \end{aligned}$$

From here:

$$\mathbf{y_2 = y_1 u = x \left( -\frac{1}{x} + x \right) = x^2 - 1.}$$

**Problem 2.** Solve the given differential equation

$$y'' + 4y' + 4y = e^{-2x}.$$

(a) By using the method of undetermined coefficients.

(b) By using the method of variation of parameters.

**Solution of (a).** Find the general solution  $y_c$  of the associated homogeneous DE:

$$\begin{aligned} y'' + 4y' + 4y = 0 &\Rightarrow m^2 + 4m + 4 = 0 \Rightarrow (m + 2)^2 = 0 \Rightarrow m_1 = m_2 = -2. \\ &\Rightarrow \mathbf{y}_c = \mathbf{c}_1 \mathbf{e}^{-2\mathbf{x}} + \mathbf{c}_2 \mathbf{x} \mathbf{e}^{-2\mathbf{x}}. \end{aligned}$$

Next, find the form of a particular solution  $y_p$  of the given differential equation. Consider the right-hand side is  $f(x) = e^{-2x}$  and differentiate it consecutively:

$$f(x) = e^{-2x}, f'(x) = -2e^{-2x}, f''(x) = 4e^{-2x}, \dots$$

to conclude that all derivatives of  $f(x)$  (including  $f(x)$ ) have the form

$$Ae^{-2x}, \quad A \text{ is a coefficient to be determined}$$

and from here, the preliminary form of the particular solution is

$$y_p(x) = Ae^{-2x}.$$

However, applying undetermined coefficients, we have to avoid the effect of overlapping of  $y_p(x)$  with  $y_c(x)$ . To ensure that there is no overlapping, we must compare  $y_c(x)$  to the preliminary form of  $y_p(x)$ . Since  $y_p = Ae^{-2x}$  matches  $e^{-2x}$  (a portion of  $y_c$ ), there is an overlapping of  $y_p(x)$  with a part of  $y_c(x)$ . Thus, you must guess another  $y_p$  by multiplying the initial  $y_p$  by  $x$ , until there is no overlapping with  $y_c$ . So, guessing  $y_p = Axe^{-2x}$ . Yet, there is still an overlapping with a part of  $y_c$ . So, guessing  $y_p = Ax^2e^{-2x}$ , there is no more overlapping, and this is the preliminary form of the particular solution:

$$\mathbf{y}_p(\mathbf{x}) = \mathbf{A} \mathbf{x}^2 \mathbf{e}^{-2\mathbf{x}}.$$

Plug-in  $y_p$  into the DE,  $y'' + 4y' + 4y = e^{-2x}$  in order to solve for  $A$ . To do this, we must first find  $y'_p$  and  $y''_p$ :

$$\begin{aligned} y_p = Ax^2e^{-2x} &\Rightarrow y'_p = 2Axe^{-2x} - 2Ax^2e^{-2x} \\ &\Rightarrow y''_p = 2Ae^{-2x} - 4Axe^{-2x} - 4Axe^{-2x} + 4Ax^2e^{-2x} \\ &\Rightarrow y''_p = (2A - 8Ax + 4Ax^2)e^{-2x} \end{aligned}$$

Plugging these values into the differential equation in order to solve for the constant  $A$ :

$$(2A - 8Ax + 4Ax^2)e^{-2x} + 4(2Ax - 2Ax^2)e^{-2x} + 4(Ax^2e^{-2x}) = e^{-2x}$$
$$\Rightarrow 2Ae^{-2x} = e^{-2x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

Thus, since we have guessed  $y_p = Ax^2e^{-2x}$  the particular solution is

$$y_p = \frac{1}{2}x^2e^{-2x}.$$

**Finally, the general solution of the given differential equation is**

$$y = y_c + y_p \quad \Rightarrow \quad y = c_1e^{-2x} + c_2xe^{-2x} + \frac{1}{2}x^2e^{-2x}.$$

**Solution of (b).** Find the general solution  $y_c$  of the associated homogeneous DE:

$$\begin{aligned} y'' + 4y' + 4y &= 0 \Rightarrow m^2 + 4m + 4 = 0 \Rightarrow (m + 2)^2 = 0 \Rightarrow m_1 = m_2 = -2. \\ \Rightarrow y_1 &= e^{-2x}, y_2 = xe^{-2x} \\ \Rightarrow \mathbf{y}_c &= \mathbf{c}_1 \mathbf{y}_1 + \mathbf{c}_2 \mathbf{y}_2 \Rightarrow \mathbf{y}_c = \mathbf{c}_1 \mathbf{e}^{-2\mathbf{x}} + \mathbf{c}_2 \mathbf{x} \mathbf{e}^{-2\mathbf{x}}. \end{aligned}$$

Next, we replace the constants  $c_1$  and  $c_2$  with functions  $u_1(x)$  and  $u_2(x)$  and we look for  $y_p$  in the form:

$$\mathbf{y}_p = \mathbf{u}_1 \mathbf{y}_1 + \mathbf{u}_2 \mathbf{y}_2 \Rightarrow \mathbf{y}_p = \mathbf{u}_1 \mathbf{e}^{-2\mathbf{x}} + \mathbf{u}_2 \mathbf{x} \mathbf{e}^{-2\mathbf{x}}.$$

In other words, we vary the parameters  $c_1$  and  $c_2$ , replacing them by functions  $u_1$  and  $u_2$  in order to find  $y_p$ .

In order to apply the method of variation of parameters first, we have to put the equation in standard form. However, the given differential equation is in standard form and we proceed by finding  $u'_1$  and  $u'_2$  with  $f(x) = e^{-2x}$ :

$$u'_1 = \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & xe^{-2x} \\ e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}} = \frac{-xe^{-4x}}{e^{-4x}} = -x.$$

$$u'_2 = \frac{W_2}{W} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}} = \frac{e^{-4x}}{e^{-4x}} = 1.$$

Then,

$$u_1 = \int -x dx = -\frac{x^2}{2}, \quad u_2 = \int 1 dx = x$$

Since  $y_p = u_1 y_1 + u_2 y_2$ ,

$$y_p = \left[ -\frac{x^2}{2} \right] [e^{-2x}] + [x] [x e^{-2x}] = \frac{1}{2} x^2 e^{-2x}.$$

Finally, the general solution of the given differential equation is

$$\mathbf{y} = \mathbf{y}_c + \mathbf{y}_p \Rightarrow \mathbf{y} = \mathbf{c}_1 \mathbf{e}^{-2\mathbf{x}} + \mathbf{c}_2 \mathbf{x} \mathbf{e}^{-2\mathbf{x}} + \frac{1}{2} \mathbf{x}^2 \mathbf{e}^{-2\mathbf{x}}.$$

**Problem 3.** Solve the following initial value problem

$$y'' + 5x^{-1}y' + 5x^{-2}y = 0, \quad x > 0.$$

$$y(1) = 0, \quad y'(1) = 1.$$

**Solution.** Looking for a solution in a form  $y = x^m$  we obtain the auxiliary quadratic:

$$m(m-1) + 5m + 5 = 0 \quad \Rightarrow \quad m^2 + 4m + 5 = 0 \quad \Rightarrow \quad m_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$\Rightarrow m_1 = -2 - i; \quad m_2 = -2 + i.$$

We have the case of complex conjugate solutions  $m_1 = -2 - i$  and  $m_2 = -2 + i$ .  
Complex fundamental set of solutions:

$$\tilde{y}_1 = x^{-2-i} = x^{-2} [\cos(\ln(x)) - i \sin(\ln(x))],$$

$$\tilde{y}_2 = x^{-2+i} = x^{-2} [\cos(\ln(x)) + i \sin(\ln(x))].$$

Real fundamental set of solutions:

$$y_1 = \mathbf{Re}(\tilde{y}_2) = x^{-2} \cos(\ln(x)), \quad y_2 = \mathbf{Im}(\tilde{y}_2) = x^{-2} \sin(\ln(x)).$$

The general solution:

$$\mathbf{y} = \mathbf{c}_1 \mathbf{x}^{-2} \cos(\ln(\mathbf{x})) + \mathbf{c}_2 \mathbf{x}^{-2} \sin(\ln(\mathbf{x})).$$

$$\mathbf{y}(\mathbf{x}) = \mathbf{x}^{-2} [\mathbf{c}_1 \cos(\ln(\mathbf{x})) + \mathbf{c}_2 \sin(\ln(\mathbf{x}))].$$

Applying the initial value conditions ( $\ln(1) = 0$ ):

$$y(1) = 0 \Rightarrow (1)^{-2} [c_1 \cos(\ln(1)) + c_2 \sin(\ln(1))] \Rightarrow \mathbf{c}_1 = \mathbf{0}$$

$$\Rightarrow y = c_2 x^{-2} \sin(\ln(x)).$$

$$\begin{aligned} y'(x) &= c_2(-2x^{-3}) \sin(\ln(x)) + c_2 x^{-2} [\ln(x)]' \cos(\ln(x)) \\ &= c_2(-2x^{-3}) \sin(\ln(x)) + c_2 x^{-3} \cos(\ln(x)) \end{aligned}$$

$$\begin{aligned} y'(1) &= c_2(-2(1)^{-3}) \sin(\ln(1)) + c_2(1)^{-3} \cos(\ln(1)) = c_2 \\ &\Rightarrow y'(1) = 1 \quad \Rightarrow \quad \mathbf{c}_2 = \mathbf{1}. \end{aligned}$$

**Finally, the unique solution of the given IVP is:**

$$\mathbf{y}(\mathbf{x}) = \mathbf{x}^{-2} \sin(\ln(\mathbf{x})) \quad \Leftrightarrow \quad \mathbf{y}(\mathbf{x}) = \frac{\sin(\ln(\mathbf{x}))}{\mathbf{x}^2}.$$

**Problem 4.** A mass weighing 16 lb stretches a spring 8 feet. The mass is initially released 2 ft below its equilibrium position with downward velocity 4 ft/sec.

(a) Formulate an initial value problem (that is: a differential equation and initial value conditions) describing the dynamics of the spring-mass system.

(b) Find an equation of the spring-mass motion.

(c) Represent the equation obtained in (b) in **an amplitude–phase-angle form**.

(d) Find the first time-moment at which the mass passes through the equilibrium position. What is the velocity of the mass at this time-moment – upward or downward?

Assume  $32 \text{ ft/sec}^2$  the acceleration due to gravity.

**Solution.** (a)

$$W = F \Rightarrow 16 = k 8 \Rightarrow \mathbf{k} = \mathbf{2}.$$

$$W = m g \Rightarrow 16 = 32 m \quad \mathbf{m} = \frac{\mathbf{1}}{\mathbf{2}} \text{ slugs.}$$

The IVP:

$$\frac{\mathbf{1}}{\mathbf{2}} \mathbf{x}''(\mathbf{t}) + \mathbf{2x}(\mathbf{t}) = \mathbf{0} \Leftrightarrow \mathbf{x}''(\mathbf{t}) + \mathbf{4x}(\mathbf{t}) = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{2}, \quad \mathbf{x}'(0) = \mathbf{4}.$$

(b) The auxiliary quadratic:  $m^2 + 4 = 0 \Rightarrow m_{1,2} = \pm i 2$ .

**Complex fundamental set of solutions (by Euler's formula):**

$$\tilde{\mathbf{x}}_1(\mathbf{t}) = \mathbf{e}^{-\mathbf{i}2\mathbf{t}} = (\text{by Euler's formula}) \cos(\mathbf{2t}) - \mathbf{i} \sin(\mathbf{2t}),$$

$$\tilde{\mathbf{x}}_2(\mathbf{t}) = \mathbf{e}^{\mathbf{i}2\mathbf{t}} = (\text{by Euler's formula}) \cos(\mathbf{2t}) + \mathbf{i} \sin(\mathbf{2t}).$$

**Real fundamental set of solutions:**

$$\mathbf{x}_1(\mathbf{t}) = \mathbf{Re} [\tilde{\mathbf{x}}_2(\mathbf{t})] = \cos(\mathbf{2t}), \quad \mathbf{x}_2(\mathbf{t}) = \mathbf{Im} [\tilde{\mathbf{x}}_2(\mathbf{t})] = \sin(\mathbf{2t}).$$

**The general solution of the differential equation:**

$$\mathbf{x}(\mathbf{t}) = \mathbf{c}_1 \cos(\mathbf{2t}) + \mathbf{c}_2 \sin(\mathbf{2t}).$$

Applying the initial value conditions:

$$x(0) = 2 \Rightarrow \mathbf{c}_1 = \mathbf{2}.$$

$$x'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) \Rightarrow x'(0) = 2c_2 = 4 \Rightarrow \mathbf{c}_2 = \mathbf{2}.$$

The equation of the given spring-mass motion:

$$\mathbf{x(t) = 2 \cos(2t) + 2 \sin(2t)}.$$

(c)

$$\begin{aligned} x(t) &= \sqrt{2^2 + 2^2} \left[ \frac{2}{\sqrt{2^2 + 2^2}} \cos(2t) + \frac{2}{\sqrt{2^2 + 2^2}} \sin(2t) \right] \\ &= 2\sqrt{2} \left[ \frac{1}{\sqrt{2}} \cos(2t) + \frac{1}{\sqrt{2}} \sin(2t) \right] \\ &= 2\sqrt{2} \left[ \sin\left(\frac{\pi}{4}\right) \cos(2t) + \cos\left(\frac{\pi}{4}\right) \sin(2t) \right] \\ &= 2\sqrt{2} \sin\left(2t + \frac{\pi}{4}\right). \end{aligned}$$

The amplitude–phase-angle representation of the equation of the spring-mass motion:

$$\mathbf{x(t) = 2\sqrt{2} \sin\left(2t + \frac{\pi}{4}\right)}.$$

(d)

$$2t^* + \frac{\pi}{4} = k\pi$$

For  $k = 0$  negative time so,  $k = 1$  and from here

$$2t^* + \frac{\pi}{4} = \pi \quad \Rightarrow \quad 2t^* = \frac{3\pi}{4} \quad \Rightarrow \quad \mathbf{t^* = \frac{3\pi}{8}}.$$

Graphically, first the motion will attain its max position, next it will attain for the first time its equilibrium position obviously with upward (negative) velocity. This can be concluded by the amplitude–phase-angle representation of the equation of the spring-mass motion:

$$\begin{aligned} x'(t) &= 4\sqrt{2} \cos\left(2t + \frac{\pi}{4}\right) \quad \Rightarrow \quad x'(t^*) = x'\left(\frac{3\pi}{8}\right) = 4\sqrt{2} \cos\left(\frac{6\pi}{8} + \frac{\pi}{4}\right) \\ &= 4\sqrt{2} \cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = 4\sqrt{2} \cos(\pi) = \mathbf{-4\sqrt{2} \text{ ft/sec}} \end{aligned}$$

hence, **upward velocity  $4\sqrt{2}$  ft/sec.**