Problem # 1 & 2

2-2.) (a) $\nabla f = \bar{a}_x \frac{\partial}{\partial x}(20x^2) = \bar{a}_x 40x.$

(b) $\nabla g = \bar{a}_x \frac{\partial(20x^2 + 30xy^2 + 40xyz)}{\partial x} + \bar{a}_y \frac{\partial(20x^2 + 30xy^2 + 40xyz)}{\partial y} + \bar{a}_z \frac{\partial(20x^2 + 30xy^2 + 40xyz)}{\partial z}$

$\bar{a}_x(40x + 30y^2 + 40yz) + \bar{a}_y(60xy + 40xz) + \bar{a}_z 40xy.$

(c) $\nabla F = \bar{a}_r \frac{\partial F}{\partial r} = \bar{a}_r \frac{\partial}{\partial r}\left(\frac{100}{r}\right) = -\bar{a}_r \frac{100}{r^2}.$

(d) $\nabla G = \bar{a}_\rho \frac{\partial}{\partial \rho}(5\rho \sin\phi - 6\rho^2 z \cos\phi) + \bar{a}_\phi \frac{\partial}{\rho\,\partial\phi}(5\rho\sin\phi - 6\rho^2 z\cos\phi) + \bar{a}_z \frac{\partial}{\partial z}(5\rho\sin\phi - 6\rho^2 z\cos\phi)$

$= \bar{a}_\rho(5\sin\phi - 12\rho z\cos\phi) + \bar{a}_\phi(5\cos\phi + 6\rho z\sin\phi) - \bar{a}_z 6\rho^2\cos\phi.$

(e) With $h = \frac{50}{r} + 10r\cos\theta + 20r^2\sin\theta\sin 2\phi$

$\nabla h = \bar{a}_r \frac{\partial h}{\partial r} + \bar{a}_\theta \frac{\partial h}{r\,\partial\theta} + \bar{a}_\phi \frac{1}{r\sin\theta}\frac{\partial h}{\partial\phi}$

$= \bar{a}_r\left(-\frac{50}{r^2} + 10\cos\theta + 40r\sin\theta\sin 2\phi\right) + \bar{a}_\theta\left(-10\sin\theta + 20r\cos\theta\sin 2\phi\right) + \bar{a}_\phi 40r\cos 2\phi.$

2-8.) (a) $\operatorname{div}\bar{A} = 0$

(b) With $F_x = 3xz$, $F_y = 4xy$, $F_z = 5x^2 + y$:

$\operatorname{div}\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

$= \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(4xy) + \frac{\partial}{\partial z}(5x^2 + y)$

$= 3z + 4x.$

(c) $\operatorname{div}\bar{G} = 0$

(d) With $H_r = 6r$, $H_\theta = 0$, $H_\phi = 0$:

$\operatorname{div}\bar{H} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 H_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot 6r) =$

$= \frac{6}{r^2}\cdot 3r^2 = 18.$

With $H_x = 6x$, $H_y = 6y$, $H_z = 6z$:

$\operatorname{div}\bar{H} = \frac{\partial(6x)}{\partial x} + \frac{\partial(6y)}{\partial y} + \frac{\partial(6z)}{\partial z}$

$= 6 + 6 + 6 = 18.$ (Checks)

(e) With $J_\rho = 5\rho z\sin\phi$, $J_\phi = 10\rho z\cos\phi$, $J_z = 0$:

$\operatorname{div}\bar{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho) + \frac{1}{\rho}\frac{\partial J_\phi}{\partial\phi}$

$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot 5\rho z\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(10\rho z\cos\phi)$

$= 10z\sin\phi - 10z\sin\phi = 0.$

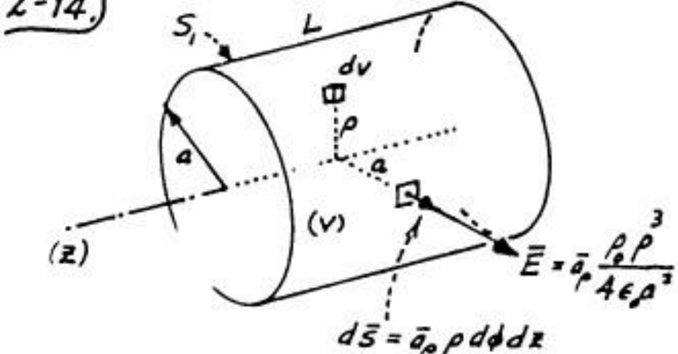
($\bar{J}$ is divergenceless and thus is sourceless.)

(f) $\operatorname{div}\bar{K} = \frac{1}{r^2}\frac{\partial}{\partial r}(100) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{20\sin\theta}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(10r\cos\phi)$

$= \frac{20}{r^2}\frac{\cos\theta}{\sin\theta} - \frac{10\sin\phi}{\sin\theta}$

Problem # 3 & 4

2-14.)



To show: that the divergence theorem (2-34) is satisfied by the given field:

$$\bar E = \bar a_\rho \frac{\rho_0 \rho^3}{4\epsilon_0 a^2} \quad \ldots (1)$$

within and on the right circular cylinder shown (radius a, length L).

(1) The closed-surface integral of (2-34) need be evaluated only on the peripheral surface S_1 shown, since $\bar E$ has only a radial component. Thus, on S_1 where $\rho = a$:

$$\oint_S \bar E \cdot d\bar s = \int_{S_1} \bar E \cdot d\bar s = \iint \frac{\rho_0 \rho^3}{4\epsilon_0 a^2} \underbrace{\bar a_\rho \cdot \bar a_\rho}_{1} \rho\, d\phi\, dz \Big]_{\rho = a}$$

$$= \frac{\rho_0 a^2}{4\epsilon_0} \int_0^{2\pi} d\phi \int_0^L dz = \frac{\pi L \rho_0 a^2}{2\epsilon_0} \quad \ldots (2)$$

(2) The volume integral $\int_V \operatorname{div} \bar E\, dv$ of (2-34) requires determining $\operatorname{div} \bar E$ first:

$$\operatorname{div} \bar E = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\rho_0 \rho^4}{4\epsilon_0 a^2}\right) = \frac{\rho_0 \rho^2}{\epsilon_0 a^2}.$$

Then,

$$\int_V \operatorname{div} \bar E\, dv = \frac{\rho_0}{\epsilon_0 a^2} \iiint \rho^2 \cdot \rho\, d\rho\, d\phi\, dz$$

$$= \frac{\rho_0}{\epsilon_0 a^2} \int_0^L dz \int_0^{2\pi} d\phi \int_0^a \rho^3 d\rho = \frac{\pi L \rho_0 a^2}{2\epsilon_0} \quad \ldots (3)$$

which checks (2); so (2-34) is satisfied.

2-22.) (a) curl $\bar A = 0$, by inspection.

(b)

$$\operatorname{curl} \bar F = \begin{vmatrix} \bar a_x & \bar a_y & \bar a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 4xy & (5x^2+y) \end{vmatrix}$$

$$= \bar a_x\left[\frac{\partial}{\partial y}(5x^2+y) - 0\right] + \bar a_y\left[\frac{\partial}{\partial z}3xz - \frac{\partial}{\partial x}(5x^2+y)\right] + \bar a_z\left[\frac{\partial}{\partial x}(4xy) - 0\right]$$

$$= \bar a_x - 7x\,\bar a_y + 4y\,\bar a_z$$

(c)

$$\nabla \times \bar G = \begin{vmatrix} \bar a_x & \bar a_y & \bar a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 4z & (5x^2+y) \end{vmatrix}$$

$$= \bar a_x\left[\frac{\partial(5x^2+y)}{\partial y} - \frac{\partial(4z)}{\partial z}\right] + \bar a_y\left[0 - \frac{\partial(5x^2+y)}{\partial x}\right] + \bar a_z\left[0 - \frac{\partial(3y)}{\partial y}\right]$$

$$= -3\bar a_x - 10x\,\bar a_y - 3\bar a_z.$$

(d) $\nabla \times \bar H = 0$, by inspection.

(e)

$$\nabla \times \bar J = \begin{vmatrix} \frac{\bar a_\rho}{\rho} & \bar a_\phi & \frac{\bar a_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 5\rho z \sin\phi & \rho \cdot 10\rho z \cos\phi & 0 \end{vmatrix}$$

$$= \frac{\bar a_\rho}{\rho}(-10\rho^2 \cos\phi) + \bar a_\phi(5\rho \sin\phi) + \frac{\bar a_z}{\cancel{\rho}}(20\rho z \cos\phi - 5\rho z \cos\phi)$$

$$= -\bar a_\rho\, 10\rho \cos\phi + \bar a_\phi\, 5\rho \sin\phi + \bar a_z\, 15 z \cos\phi$$

(f)

$$\nabla \times \bar K = \begin{vmatrix} \frac{\bar a_r}{r^2 \sin\theta} & \frac{\bar a_\theta}{r \sin\theta} & \frac{\bar a_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{100}{r^2} & r \cdot \left(\frac{20}{r}\right) & (r \sin\theta) \cdot 10 r \cos\phi \end{vmatrix}$$

$$= \frac{\bar a_r}{r^2 \sin\theta}\left[\frac{\partial}{\partial \theta}(10 r^2 \sin\theta \cos\phi) - 0\right]$$

$$+ \frac{\bar a_\theta}{r \sin\theta}\left[0 - \frac{\partial}{\partial r}(10 r^2 \sin\theta \cos\phi)\right]$$

$$+ \frac{\bar a_\phi}{r}[0 - 0]$$

$$= \bar a_r\, 10 \cot\theta \cos\phi - \bar a_\theta\, 20 \cos\phi$$

Problem # 5

2-29.) (See the given diagram.)

(a) Using (2-51) in cylindrical coordinates

$$\text{curl } \bar{E} = \begin{vmatrix} \frac{\bar{a}_\rho}{\rho} & \bar{a}_\phi & \frac{\bar{a}_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 5\rho z & -8\rho z^2 & 10z^2 \sin\phi \end{vmatrix} = \frac{\bar{a}_\rho}{\rho}\left[10z^2\cos\phi + 16\rho z\right] + \bar{a}_\phi 5\rho - \frac{\bar{a}_z}{\rho} 8z^2$$

$$= \bar{a}_\rho\left(\frac{10z^2}{\rho}\cos\phi + 16z\right) + \bar{a}_\phi 5\rho - \bar{a}_z \frac{8z^2}{\rho}.$$

Since curl $\bar{E} \neq 0$, $\bar{E}$ is not a conservative field.

(b) Choose the positive sense of the line integration about ℓ in the counterclockwise sense, looking at S. Then

$$\oint_\ell \bar{E}\cdot d\bar{\ell} = \int_{\ell_1} \bar{E}\cdot d\bar{\ell} + \int_{\ell_2} \bar{E}\cdot d\bar{\ell} + \int_{\ell_3} \bar{E}\cdot d\bar{\ell} + \int_{\ell_4} \bar{E}\cdot d\bar{\ell}$$

with

$$\int_{\ell_1} \bar{E}\cdot d\bar{\ell} = \int_{\ell_1} (-\bar{a}_\phi 8z^2)\cdot \bar{a}_\phi \rho\, d\phi \Big]_{\substack{\rho=2\\ z=0}} = 0 \text{ (since } z=0 \text{ on } \ell_1)$$

$$\int_{\ell_2} \bar{E}\cdot d\bar{\ell} = \int_{z=0}^{3} (\bar{a}_z 10z^2 \sin\phi)\cdot \bar{a}_z\, dz \Big]_{\substack{\rho=2\\ \phi=\pi/2}} = 10\int_0^3 z^2\, dz = 90.$$

$$\int_{\ell_3} \bar{E}\cdot d\bar{\ell} = \int_{\phi=\pi/2}^{0} (-\bar{a}_\phi 8z^2)\cdot \bar{a}_\phi \rho\, d\phi \Big]_{\substack{\rho=2\\ z=3}} = -144\int_{\pi/2}^{0} d\phi = 72\pi.$$

$$\int_{\ell_4} \bar{E}\cdot d\bar{\ell} = \int_{\ell_4} (\bar{a}_z 10z^2 \sin\phi)\cdot \bar{a}_z\, dz \Big]_{\substack{\rho=2\\ \phi=0}} = 0 \text{ (since } \phi=0 \text{ on } \ell_4)$$

making

$$\oint_\ell \bar{E}\cdot d\bar{\ell} = 90 + 72\pi = 316.2$$

(c) Verify the answer to (b) using Stokes's theorem (2-56), employing $\int_S (\text{curl } \bar{E})\cdot d\bar{s}$ this time. The correct sense of $d\bar{s}$ on S is decided using the right hand rule. With the fingers of the right hand pointing in the positive line-integration sense about ℓ chosen in (b), the thumb points in the positive $\bar{a}_\rho$ sense out of S, so that $d\bar{s} = \bar{a}_\rho\, ds = \bar{a}_\rho\, \rho\, d\phi\, dz$ on S.

Using the curl $\bar{E}$ result of (a):

$$\int_S (\text{curl } \bar{E})\cdot d\bar{s} = \int_S \left[\bar{a}_\rho\left(\frac{10z^2}{\rho}\cos\phi + 16z\right)\right]\cdot \bar{a}_\rho \rho\, d\phi\, dz \Big]_{\rho=2}$$

$$= \int_{z=0}^{3}\int_{\phi=0}^{\pi/2} (5z^2\cos\phi + 16z)\, 2\, d\phi\, dz$$

$$= \int_0^3\int_0^{\pi/2} 10z^2\cos\phi\, d\phi\, dz + \int_0^3\int_0^{\pi/2} 32z\, d\phi\, dz$$

$$= 10\int_0^3 z^2 dz \int_0^{\pi/2}\cos\phi\, d\phi + 32\int_0^3 z\, dz\int_0^{\pi/2} d\phi$$

$$= 10\left[\sin\phi\right]_0^{\pi/2}\left[\frac{z^3}{3}\right]_0^3 + 32\left(\frac{\pi}{2}\right)\frac{9}{2}$$

$$= 90 + 72\pi = 316.2$$

which checks the closed-line integral result of (b).

Problem # 6

2·31.) (a) Given $\bar{B} = \bar{a}_y \mu_o J_z x$, then

$$\nabla \times \frac{\bar{B}}{\mu_o} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & J_z x & 0 \end{vmatrix} = \bar{a}_z \frac{\partial}{\partial x}(J_z x) = \bar{a}_z J_z$$

which is precisely the current density $\bar{J}$ in the conducting slab; thus (2·65) is satisfied.

(b) Given the $\bar{B}$-field in the hollow conductor

$$\bar{B} = \bar{a}_\phi \frac{\mu_o I(\rho^2 - b^2)}{2\pi\rho(c^2 - b^2)}$$

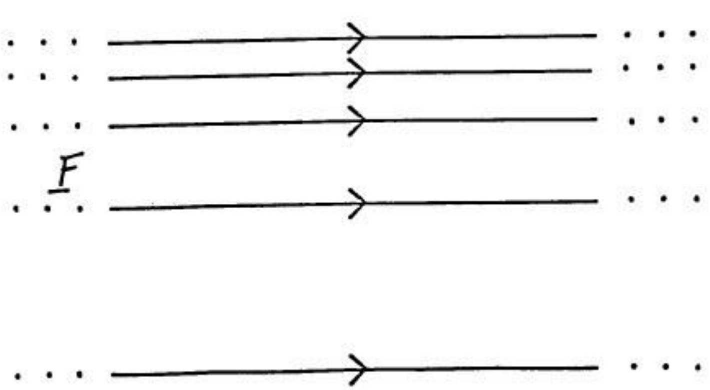
then

$$\nabla \times \frac{\bar{B}}{\mu_o} = \begin{vmatrix} \frac{\bar{a}_\rho}{\rho} & \bar{a}_\phi & \frac{\bar{a}_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho \cdot \frac{I(\rho^2-b^2)}{2\pi\rho(c^2-b^2)} & 0 \end{vmatrix}$$

$$= \frac{\bar{a}_z}{\rho} \frac{\partial}{\partial \rho} \frac{I(\rho^2-b^2)}{2\pi(c^2-b^2)} = \bar{a}_z \frac{I}{2\pi(c^2-b^2)} 2\rho$$

$$= \bar{a}_z \frac{I}{\pi(c^2-b^2)},$$ the current density $\bar{J}$ in that conductor; so (2·65) is satisfied.

Problem # 7



Assuming RECTANGULAR CO-ORDS.:

$\nabla \cdot \underline{F} = 0$; $\nabla \times \underline{F} \neq 0$

FOR STATIC SYSTEMS:

$\nabla \times \underline{E} = 0$; $\nabla \cdot \underline{E} = \frac{\rho_v}{\varepsilon_o}$.

$\nabla \times \underline{B} = \underline{J}$; $\nabla \cdot \underline{B} = 0$.

$\Rightarrow$ $\underline{B}$ - FIELD!