Assignment # 7

1a) $|\psi\rangle = A\left[6\,|4\,2\,{-2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - 8\,|3\,1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right]$

$\langle\psi|\psi\rangle = 1$

$A^2\left[6\langle 4\,2\,{-2}|\langle\tfrac{1}{2}\,\tfrac{1}{2}| - 8\langle 3\,1\,1|\langle\tfrac{1}{2}\,{-\tfrac{1}{2}}|\right]\left[6\,|4\,2\,{-2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - 8\,|3\,1\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\right] = 1$

$A^2\,(36 + 64) = 1$

$A = \tfrac{1}{10}$

b) total orbital $\quad l = 2$ so $6\hbar^2$ with $\left|\tfrac{6}{10}\right|^2 = 36\%$ chance

or

$l = 1$ so $2\hbar^2$ with $\left|\tfrac{-8}{10}\right|^2 = 64\%$ chance

z-component orbital $\quad m_\ell = -2$ so $-2\hbar$ $\quad 36\%$

or $m_\ell = 1$ so $\hbar$ $\quad 64\%$

c) total spin $\quad s = \tfrac{1}{2}$ for electron so $\tfrac{3}{4}\hbar^2$ $\quad 100\%$

z-component spin $\quad m_s = +\tfrac{1}{2}$ $\quad \tfrac{\hbar}{2}$ $\quad 36\%$

or $m_s = -\tfrac{1}{2}$ $\quad -\tfrac{\hbar}{2}$ $\quad 64\%$

d) $$\langle \hat{r}^n\rangle = \int_0^\infty\int_0^\pi\int_0^{2\pi} \psi^* r^n \psi\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^\infty \left(\frac{4}{81\sqrt{30}}\right)^2 \frac{1}{a^3}\,\frac{1}{a^4}\, r^{n+6}\, e^{-2r/3a}\, dr$$

$$= \frac{16}{196830}\,\frac{1}{a^7}\,\frac{\Gamma(n+7)}{\left(\frac{2}{3a}\right)^{n+7}}$$

$$= \frac{16}{196830}\left(\frac{3}{2}\right)^{n+7}(n+6)!\,a^n$$

2 a) $|\psi\rangle = |4\,3\,3\rangle$

$$= \left(\frac{1}{768\sqrt{35}}\, a^{-3/2} \left(\frac{r}{a}\right)^3 e^{-r/4a}\right)\left(-\sqrt{\frac{35}{64\pi}}\,\sin^3\theta\, e^{3i\phi}\right)$$

b) $$\langle \hat{r} \rangle = \int_0^\infty r \left(\frac{1}{768\sqrt{35}}\right)^2 a^{-3} \left(\frac{r}{a}\right)^6 e^{-r/2a}\, r^2\, dr$$ ($Y_3^3(\theta,\phi)$ is normalized)

$$= \left(\frac{1}{768\sqrt{35}}\right)^2 \frac{1}{a^9} \int_0^\infty r^9 e^{-r/2a}\, dr$$

$$= \left(\frac{1}{768\sqrt{35}}\right)^2 \frac{1}{a^9} \frac{\Gamma(10)}{\left(\frac{1}{2a}\right)^{10}}$$

$$= \left(\frac{1}{768\sqrt{35}}\right)^2 9!\, 2^{10} a$$

$$= 13824\, a$$

c) $$\hat{L}_x^2 + \hat{L}_y^2 = \hat{\vec{L}}^2 - \hat{L}_z^2$$

So you would measure $12\hbar^2 - 9\hbar^2 = 3\hbar^2$

3 a) $\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = -\frac{e}{m} B_0 \hat{S}_z \qquad \vec{B} = B_0 \hat{k}$

$$= \frac{-e\hbar B_0}{2m} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $E = \frac{-e\hbar B_0}{2m}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $E = +\frac{e\hbar B_0}{2m}$

General vector $a\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\frac{ieB_0 t}{2m}} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\frac{ieB_0 t}{2m}}$ with $a^2 + b^2 = 1$

b) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\frac{ieB_0 t}{2m}} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\frac{ieB_0 t}{2m}} \right)$

at $t = 0$ $\langle \hat{S}_x \rangle = -\frac{\hbar}{2}$

in general $\langle \hat{S}_x \rangle = \frac{\hbar}{4} \left[e^{-ieB_0 t/2m} \quad e^{ieB_0 t/2m} \right] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{ieB_0 t/2m} \\ e^{-ieB_0 t/2m} \end{bmatrix}$

$$= \frac{\hbar}{4}\left(e^{-ieB_0 t/m} + e^{ieB_0 t/m} \right)$$

$$= \frac{\hbar}{2} \cos\left(\frac{eB_0 t}{m} \right)$$

So $\langle \hat{S}_x \rangle$ will be $-\frac{\hbar}{2}$ whenever $t = \frac{m}{eB_0}(2n+1)\pi$ n an integer

and the frequency is $\omega = \frac{eB_0}{m}$