Assignment #6 2015

1\. a) $[\hat{L}_z, \hat{L}_y] = [(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x), (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)]$

$= [\hat{x}\hat{p}_y, \hat{z}\hat{p}_x] - [\hat{x}\hat{p}_y, \hat{x}\hat{p}_z] - [\hat{y}\hat{p}_x, \hat{z}\hat{p}_x] + [\hat{y}\hat{p}_x, \hat{x}\hat{p}_z]$

$= \hat{z}\hat{p}_y [\hat{x}, \hat{p}_x] + \hat{y}\hat{p}_z [\hat{p}_x, \hat{x}]$

$= \hat{z}\hat{p}_y (i\hbar) + \hat{y}\hat{p}_z (-i\hbar)$

$= -i\hbar (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)$

$= -i\hbar \hat{L}_x$

b) $[\hat{L}_+, \hat{L}_-] = [(\hat{L}_x + i\hat{L}_y), (\hat{L}_x - i\hat{L}_y)]$

$= -[\hat{L}_x, i\hat{L}_y] + [i\hat{L}_y, \hat{L}_x]$

$= -i(i\hbar \hat{L}_z) + i(-i\hbar \hat{L}_z)$

$= 2\hbar \hat{L}_z$

d) $\hat{L}_x^{\dagger} = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)^{\dagger}$

$= (\hat{y}\hat{p}_z)^{\dagger} - (\hat{z}\hat{p}_y)^{\dagger}$

$= \hat{p}_z^{\dagger} \hat{y}^{\dagger} - \hat{p}_y^{\dagger} \hat{z}^{\dagger}$

$= \hat{p}_z \hat{y} - \hat{p}_y \hat{z}$ (since all Hermitian operators)

$= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$

$= \hat{L}_x$

Similarly for $\hat{L}_y$

1 c) $x = r\sin\theta\cos\phi$ $\quad \frac{\partial}{\partial x} = \cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}$

$y = r\sin\theta\sin\phi$ $\quad \frac{\partial}{\partial y} = \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta}$

$z = r\cos\theta$

$\hat{L}_z = \hat{x}\hat{P}_y - \hat{y}\hat{P}_x$

$= -i\hbar x\frac{\partial}{\partial y} + i\hbar y\frac{\partial}{\partial x}$

$= -i\hbar\left(r\sin\theta\cos\phi\right)\left(\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$

$+ i\hbar\left(r\sin\theta\sin\phi\right)\left(\cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$

$= -i\hbar\left(\cos^2\phi + \sin^2\phi\right)\frac{\partial}{\partial\phi}$

$= -i\hbar\frac{\partial}{\partial\phi}$

Hilroy

2 a) $\hat{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle$

$\hat{L}_z |lm\rangle = m\hbar |lm\rangle$

$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$

$\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$

$\langle \hat{L}_x^2 \rangle = \frac{1}{4}\langle \hat{L}_+\hat{L}_+ + \hat{L}_-\hat{L}_- + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+ \rangle$

$= \frac{1}{4}\langle \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+ \rangle$ since in eigenstate $|lm\rangle$

$\langle \hat{L}_y^2 \rangle = -\frac{1}{4}\langle \hat{L}_+\hat{L}_+ + \hat{L}_-\hat{L}_- - \hat{L}_+\hat{L}_- - \hat{L}_-\hat{L}_+ \rangle$

$= \frac{1}{4}\langle \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+ \rangle$

$= \langle \hat{L}_x \rangle^2$

then $\langle \hat{L}_x^2 \rangle = \langle \hat{L}^2 - \hat{L}_y^2 - \hat{L}_z^2 \rangle$

$2\langle \hat{L}_x^2 \rangle = \langle \hat{L}^2 - \hat{L}_z^2 \rangle$

$\langle \hat{L}_x^2 \rangle = \frac{1}{2}\left[\hbar^2 l(l+1) - m^2\hbar^2\right]$

for $m = \pm l$ $\quad \langle \hat{L}_x^2 \rangle = \frac{1}{2}\hbar^2 l$

for $m = 0$ $\quad \langle \hat{L}_x^2 \rangle = \frac{1}{2}\hbar^2 l(l+1)$

$\therefore \quad \frac{1}{2}\hbar^2 l \leq \langle \hat{L}_x^2 \rangle \leq \frac{1}{2}\hbar^2 l(l+1)$

3. $\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right] \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

a) could measure $\frac{\hbar}{2}$ with $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ or 50% chance

or $-\frac{\hbar}{2}$ with $\left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ " "

b) $\hat{S}_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ $\langle \hat{S}_z\rangle = \frac{1}{\sqrt{2}}[1\ -1]\frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$= \frac{\hbar}{4}[1\ -1]\begin{bmatrix}1\\1\end{bmatrix}$

$= 0$ makes sense because 50% measure as spin up and 50% spin down

$\hat{S}_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $\hat{S}_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$

$\langle \hat{S}_x\rangle = \frac{\hbar}{4}[1\ -1]\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = -\frac{\hbar}{2}$

$\langle \hat{S}_x^2\rangle = \frac{\hbar^2}{8}[1\ -1]\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{\hbar^2}{4}$

$\langle \hat{S}_y\rangle = \frac{\hbar}{4}[1\ -1]\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$

$\langle \hat{S}_y^2\rangle = \frac{\hbar^2}{8}[1\ -1]\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{\hbar^2}{4}$

$\sigma_{S_x}^2\sigma_{S_y}^2 = \left(\frac{\hbar^2}{4} - \frac{\hbar^2}{4}\right)\left(\frac{\hbar^2}{4} - 0\right) = 0$

But! $\sigma_{S_x}^2\sigma_{S_y}^2 \geq \left[\frac{1}{2i}\langle[\hat{S}_x, \hat{S}_y]\rangle\right]^2$

$\geq \left[\frac{i\hbar}{2i}\langle \hat{S}_z\rangle\right]^2 = 0$ so ok!