Assignment #4 2015

1 a) $$\hbar\omega\left[\hat{a}_+\hat{a}_- + \tfrac{1}{2}\right] = \hbar\omega\left[\frac{1}{2\hbar m\omega}\left(-i\hat{p}+m\omega\hat{x}\right)\left(i\hat{p}+m\omega\hat{x}\right)+\tfrac{1}{2}\right]$$

$$= \hbar\omega\left[\frac{1}{2\hbar m\omega}\left[\hat{p}^2 + m^2\omega^2\hat{x}^2 + im\omega\left(\hat{x}\hat{p}-\hat{p}\hat{x}\right)\right]+\tfrac{1}{2}\right]$$

$$= \hbar\omega\left[\frac{1}{2\hbar m\omega}\left[\hat{p}^2 + m^2\omega^2\hat{x}^2 + im\omega\left(i\hbar\right)\right]+\frac{1}{2}\right]$$

$$= \hbar\omega\left[\frac{\hat{p}^2}{2\hbar m\omega} + \frac{m\omega\hat{x}^2}{2\hbar} - \frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$= \hat{H}$$

b) $$\left[\hat{a}_+, \hat{a}_-\right] = \frac{1}{2\hbar m\omega}\left[\left(-i\hat{p}+m\omega\hat{x}\right)\left(i\hat{p}+m\omega\hat{x}\right) - \left(i\hat{p}+m\omega\hat{x}\right)\left(-i\hat{p}+m\omega\hat{x}\right)\right]$$

$$= \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + m^2\omega^2\hat{x}^2 + im\omega\left(\hat{x}\hat{p}-\hat{p}\hat{x}\right) - \hat{p}^2 - m^2\omega^2\hat{x}^2 + im\omega\left(\hat{x}\hat{p}-\hat{p}\hat{x}\right)\right]$$

$$= \frac{i}{2\hbar}\left(2i\hbar\right)$$

$$= -1$$

1. c) $\hat{H}(\hat{a}_+|\psi\rangle) = \hbar\omega\left[\hat{a}_+\hat{a}_- + \tfrac{1}{2}\right]\hat{a}_+|\psi\rangle$

$= \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\hat{a}_+\right)|\psi\rangle$

$= \hat{a}_+\hbar\omega\left[\hat{a}_-\hat{a}_+ - \tfrac{1}{2} + 1\right]|\psi\rangle$

$= \hat{a}_+\left[\hat{H} + \hbar\omega\right]|\psi\rangle$

$= \hat{a}_+\left[E + \hbar\omega\right]|\psi\rangle$

$= \left[E + \hbar\omega\right]\hat{a}_+|\psi\rangle$

2. $\hat{a}_+ |n\rangle = c\, |n+1\rangle$ solve for c

So $\langle n | a_+^\dagger a_+ | n\rangle = \langle n | \hat{a}_- \hat{a}_+ | n\rangle$

$$|c|^2 \langle n+1 | n+1\rangle = \langle n | \frac{\hat{H}}{\hbar\omega} + \tfrac{1}{2} | n\rangle$$

$$|c|^2 = \langle n | \frac{(n+\frac{1}{2})\hbar\omega}{\hbar\omega} + \tfrac{1}{2} | n\rangle$$

$$= n+1$$

$$c = \sqrt{n+1}$$

Similarly for $\hat{a}_- |n\rangle$

Then

$$\langle n | \hat{x} | n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle n | \hat{a}_+ + \hat{a}_- | n\rangle$$

$= 0$ (since states end up orthogonal

Similarly $\langle \hat{p} \rangle = 0$

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega} \langle n | (\hat{a}_+ + \hat{a}_-)^2 | n\rangle$$

$$= \frac{\hbar}{2m\omega} \left[\sqrt{n}\sqrt{n} \langle n | n\rangle + \sqrt{n+1}\sqrt{n+1} \langle n | n\rangle \right]$$

$$= \frac{\hbar}{2m\omega} (2n+1)$$

$$\langle \hat{p}^2 \rangle = -\frac{\hbar m\omega}{2} \left[\langle n | \hat{a}_+^2 | n\rangle + \langle n | \hat{a}_-^2 | n\rangle - \langle n | \hat{a}_+ \hat{a}_- | n\rangle - \langle n | \hat{a}_- \hat{a}_+ | n\rangle \right]$$

$$= \frac{\hbar m\omega}{2} (2n+1)$$

So $\sigma_{\hat{x}} \sigma_{\hat{p}} = \sqrt{\frac{\hbar}{2m\omega}(2n+1) \frac{\hbar m\omega}{2}(2n+1)} = (2n+1)\frac{\hbar}{2} \geq \frac{\hbar}{2}$

$\psi(x,0) = 12\psi_0 + 5\psi_3$

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a) $A^*\left(12\langle\psi_0| + 5\langle\psi_3|\right) A\left(12|\psi_0\rangle + 5|\psi_3\rangle\right) = 1$

$A^2\left[144\langle\psi_0|\psi_0\rangle + 25\langle\psi_3|\psi_3\rangle + 60\langle\psi_0|\psi_3\rangle + 60\langle\psi_3|\psi_0\rangle\right] = 1$

$A^2\left[144 + 25\right] = 1$

$A = 1/13$

b) $\psi(x,t) = \frac{1}{13}\left[12\psi_0 e^{-i\frac{1}{2}\omega t} + 5\psi_3 e^{-i\frac{7}{2}\omega t}\right]$

$|\psi|^2 = \frac{1}{169}\left[144\psi_0^2 + 25\psi_3^2 + 60\psi_0\psi_3 e^{-6i\omega t} + 60\psi_3\psi_0 e^{+6i\omega t}\right]$

$= \frac{1}{169}\left[144\psi_0^2 + 25\psi_3^2 + 120\psi_0\psi_3 \cos 6\omega t\right]$

c) $E_0 = \frac{1}{2}\hbar\omega \qquad P = \left|\frac{12}{13}\right|^2 = \frac{144}{169} \qquad 85.2\%$

$E_3 = \frac{7}{2}\hbar\omega \qquad P = \left|\frac{5}{13}\right|^2 = \frac{25}{169} \qquad 14.8\%$

d) For particles measured in the G.S. they are then in $\psi_0 = \left[\frac{m\omega}{\pi\hbar}\right]^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$

So calculate $\int_0^1 \left(\left[\frac{m\omega}{\pi\hbar}\right]^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} dx\right.$

$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_0^1 e^{-\frac{m\omega x^2}{\hbar}} dx$

$= \sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\pi\hbar}{4m\omega}} \left[\operatorname{erf}\left(\sqrt{\frac{m\omega}{\hbar}}\right) - \operatorname{erf}(0)\right]$

$= \frac{1}{2}\operatorname{erf}\left(\sqrt{\frac{m\omega}{\hbar}}\right)$