Assignment # 3

1 a) $[\hat{A}+\hat{B}, \hat{C}] = (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B})$
$= \hat{A}\hat{C} + \hat{B}\hat{C} - \hat{C}\hat{A} - \hat{C}\hat{B}$
$= \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B}$
$= [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$

b) $[\hat{A}\hat{B}, \hat{C}] = (\hat{A}\hat{B})\hat{C} - \hat{C}(\hat{A}\hat{B})$
$= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{A}\hat{C}\hat{B}$
$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B}$
$= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

c) $\hat{A}\hat{B} = \sum_j A_{ij} B_{jk}$

Now $(\hat{A}\hat{B})^\dagger = (\sum A_{ij} B_{jk})^\dagger$ ($\dagger \equiv$ hermitian conjugate)
$= (\sum A_{ij}^* B_{jk}^*)^T$ ($T \equiv$ transpose)
$= \sum_j A_{ji}^* B_{kj}^*$
$= \sum_j B_{kj}^* A_{ji}^*$
$= \hat{B}^\dagger \hat{A}^\dagger$

d) $\sigma_{\hat{A}\hat{B}}^2 \sigma_{\hat{C}}^2 \geq \left[\frac{1}{2i}\langle[\hat{A}\hat{B}, \hat{C}]\rangle\right]^2$
$\geq \left[\frac{1}{2i}\langle\hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}\rangle\right]^2$
$\geq \left[\frac{1}{2i}\langle\hat{A}^2 + \hat{B}^2\rangle\right]^2$

e) $[\hat{x}^2, \hat{p}^2] f(x) = x^2\left(-i\hbar\frac{\partial}{\partial x}\right)^2 f(x) - \left(-i\hbar\frac{\partial}{\partial x}\right)^2 x^2 f(x)$
$= -x^2\hbar^2\frac{\partial^2 f(x)}{\partial x^2} + \hbar^2 x^2\frac{\partial^2 f(x)}{\partial x^2} + 2\hbar^2 f(x) + 4\hbar^2 x\frac{\partial f(x)}{\partial x}$
$= \left(2\hbar^2 + 4\hbar^2 x\frac{\partial}{\partial x}\right) f(x)$
$= (2\hbar^2 + 4i\hbar\hat{x}\hat{p}) f(x)$

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1e) Alternately $[\hat{x}^2, \hat{p}^2] = \hat{x}[\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2]\hat{x}$

$$= \hat{x}\hat{p}[\hat{x}, \hat{p}] + \hat{x}[\hat{x}, \hat{p}]\hat{p} + \hat{p}[\hat{x}, \hat{p}]\hat{x} + [\hat{x}, \hat{p}]\hat{p}\hat{x}$$

$$= \hat{x}\hat{p}\, i\hbar + \hat{x}\hat{p}\, i\hbar + \hat{p}\, i\hbar\, \hat{x} + i\hbar\, \hat{p}\hat{x}$$

$$= 2\hat{x}\hat{p}\, i\hbar + 2\hat{p}\hat{x}\, i\hbar$$

$$= 2\hat{x}\hat{p}\, i\hbar + 2i\hbar(\hat{x}\hat{p} - i\hbar)$$

$$\boxed{= 4i\hbar\hat{x}\hat{p} + 2\hbar^2}$$

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Scherrer Chapter 5 # 6 recall $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$

As in class:

$$i\hbar \frac{\partial}{\partial t}\langle \hat{Q}\rangle = i\hbar \frac{\partial}{\partial t}\langle \Psi | \hat{Q} | \Psi \rangle$$

$$= i\hbar \int_{-\infty}^{\infty} \left[\frac{\partial \Psi^*}{\partial t}\hat{Q}\Psi + \Psi^*\hat{Q}\frac{\partial \Psi}{\partial t}\right]dx + i\hbar\left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$= i\hbar \int_{-\infty}^{\infty} \left[\left(\frac{1}{-i\hbar}\right)\hat{H}\Psi^*\hat{Q}\Psi + \Psi^*\hat{Q}\left(\frac{1}{i\hbar}\right)\hat{H}\Psi\right]dx + i\hbar\left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

$$= \int_{-\infty}^{\infty} \left[-\Psi^*\hat{H}\hat{Q}\Psi + \Psi^*\hat{Q}\hat{H}\Psi\right]dx + i\hbar\left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$ since $\hat{H}$ hermitian

$$= \int_{-\infty}^{\infty} \Psi^*[\hat{Q},\hat{H}]\Psi\, dx + \underbrace{i\hbar\left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle}_{\text{zero since } \hat{Q} \text{ not time dependent}}$$

$$= \langle [\hat{Q},\hat{H}]\rangle$$

3

a) Suppose $\hat{O}$ is Hermitian and $|v\rangle$ is an eigenvector of $\hat{O}$

then $\hat{O}|v\rangle = \lambda|v\rangle$

So
$$\langle v|\hat{O}|v\rangle = \langle v|\hat{O}^{+}|v\rangle$$
$$\langle v|\lambda|v\rangle = (|v\rangle^{+}\hat{O}^{+})\,|v\rangle$$
$$\lambda\langle v|v\rangle = (\hat{O}|v\rangle)^{+}\,|v\rangle$$
$$\lambda = (\lambda|v\rangle)^{+}\,|v\rangle$$
$$\lambda = \langle v|\lambda^{*}|v\rangle$$
$$\lambda = \lambda^{*}\langle v|v\rangle$$
$$\lambda = \lambda^{*} \quad \therefore \ \lambda \text{ is real}$$

b) Suppose $\hat{O}|v\rangle = \lambda|v\rangle$ and $\hat{O}|w\rangle = \beta|w\rangle$ for $\lambda \neq \beta$ and both real

Now
$$\langle v|\hat{O}^{+}|w\rangle = \langle v|\hat{O}|w\rangle$$
$$(\hat{O}|v\rangle)^{+}|w\rangle = \beta\langle v|w\rangle$$
$$(\lambda|v\rangle)^{+}|w\rangle = \beta\langle v|w\rangle$$
$$\lambda^{*}\langle v|w\rangle = \beta\langle v|w\rangle$$
$$(\lambda-\beta)\langle v|w\rangle = 0$$

since $\lambda \neq \beta$ then $\langle v|w\rangle = 0$ i.e. they are orthogonal

4. Show $\hat{x}^+ = \hat{x}$

look at $\langle \psi_1 | \hat{x} | \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^* x \psi_2 \, dx$

$= \int_{-\infty}^{\infty} \psi_1^* x^* \psi_2 \, dx$ (since x real)

$= \int_{-\infty}^{\infty} x^* \psi_1^* \psi_2 \, dx$ (since just functions)

$= \int_{-\infty}^{\infty} (x \psi_1)^* \psi_2 \, dx$

$= \langle \hat{x}^+ \rangle$ $\therefore$ $\hat{x}$ is Hermitian

b) $\langle \psi_1 | \hat{p} | \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi_2 \, dx$

$= \int_{-\infty}^{\infty} i\hbar \frac{\partial}{\partial x} \psi_1^* \psi_2 \, dx + (-i\hbar) \psi_1^* \psi_2 \Big|_{-\infty}^{\infty}$ integration by parts

$= \int_{-\infty}^{\infty} \left(-i\hbar \frac{\partial}{\partial x} \psi_1\right)^* \psi_2 \, dx$ $\psi_1, \psi_2 \to 0$ at $x \to \pm\infty$

$= \langle \hat{p}^+ \rangle$ $\therefore$ $\hat{p}$ is Hermitian

c) either do integration by parts twice like in b) or argue:

① suppose $\hat{A}$ and $\hat{B}$ are hermitian and $\hat{C} = \hat{A} + \hat{B}$

then $\langle \psi_1 | \hat{C} | \psi_2 \rangle = \langle \psi_1 | (\hat{A} + \hat{B}) | \psi_2 \rangle$

$= \langle \psi_1 | \hat{A}^+ + \hat{B}^+ | \psi_2 \rangle$

$= \langle \psi_1 | (\hat{A} + \hat{B})^+ | \psi_2 \rangle$

$= \langle \hat{C}^+ \rangle$ so $\hat{C}$ also hermitian

② $\hat{C} = \hat{A}^2$ then $\langle \psi_1 | \hat{C} | \psi_2 \rangle = \langle \psi_1 | \hat{A}\hat{A} | \psi_2 \rangle$

$= \langle \psi_1 | \hat{A}^+ \hat{A}^+ | \psi_2 \rangle$

$= \langle \psi_1 | (\hat{A}\hat{A})^+ | \psi_2 \rangle$

$= \langle \hat{C}^+ \rangle$ $\therefore$ $\hat{C}$ also hermitian

Since $\hat{H}$ is a simple fnc. of operators $\hat{p}$ and $\hat{x}$ and given ① & ②

then $\hat{H}$ is also hermitian