Assignment #2

1 a) Let $\psi(x) = a\psi_1(x) + b\psi_2(x)$

$$\hat{T}_1 \psi(x) = x^3[a\psi_1(x) + b\psi_2(x)]$$
$$= a x^3 \psi_1(x) + b x^3 \psi_2(x)$$
$$= a\hat{T}_1\psi_1(x) + b\hat{T}_1\psi_2(x) \qquad \text{linear}$$

b) $$\hat{T}_2\psi(x) = x\frac{d}{dx}[a\psi_1(x) + b\psi_2(x)]$$
$$= a x\frac{d}{dx}\psi_1(x) + b x\frac{d}{dx}\psi_2(x)$$
$$= a\hat{T}_2\psi_1(x) + b\hat{T}_2\psi_2(x) \qquad \text{also linear}$$

c) $$\hat{T}_3\psi(x) = \lambda\psi^*(x)$$
$$= \lambda[a\psi_1(x) + b\psi_2(x)]^*$$
$$= \lambda[a^*\psi_1^*(x) + b^*\psi_2^*(x)]$$
$$= a^*\hat{T}_3\psi_1(x) + b^*\hat{T}_3\psi_2(x) \qquad \text{not linear}$$
($a \neq a^*$, $b \neq b^*$ in general)

d.) $$\hat{T}_4\psi(x) = e^{[a\psi_1(x) + b\psi_2(x)]}$$
$$= e^a\hat{T}_4\psi_1(x) + e^b\hat{T}_4\psi_2(x) \qquad \text{not linear}$$
($e^a \neq a$, $e^b \neq b$ in general)

e) $$\hat{T}_5\psi(x) = \frac{d}{dx}[a\psi_1(x) + b\psi_2(x)] + A$$

$$= a\hat{T}_5\psi_1(x) - A + b\hat{T}_5\psi_2(x) - A + A$$
$$= a\hat{T}_5\psi_1(x) + b\hat{T}_5\psi_2(x) - A \qquad \text{not linear}$$
(unless $A = 0$)

f) $$\hat{T}_6\psi(x) = \int_{-\infty}^{x} y[a\psi_1(y) + b\psi_2(y)]\,dy$$

$$= a\int_{-\infty}^{x} y\psi_1(y)\,dy + b\int_{-\infty}^{x} y\psi_2(y)\,dy$$

$$= a\hat{T}_6\psi_1(x) + b\hat{T}_6\psi_2(x) \qquad \text{also linear}$$

1 cont'd Find $\psi(x)$ when $\hat{T}_6 \psi(x) = \lambda \psi(x)$

$$\int_{-\infty}^{x} y\psi(y)\, dy = \lambda \psi(x)$$

differentiate both sides

$$x\psi(x) = \lambda \frac{d\psi(x)}{dx}$$

$$\frac{d\psi(x)}{\psi(x)} = \frac{x}{\lambda}\, dx$$

$$\psi(x) = A e^{\frac{x^2}{2\lambda}}$$

So $\lambda < 0$ for this to be a valid wavefunction

$$[\hat{T}_2, \hat{T}_6]\psi(x) = x\frac{d}{dx}\int_{-\infty}^{x} y\psi(y)\, dy - \int_{-\infty}^{x} y^2 \frac{d\psi(y)}{dy}\, dy$$

$$= x^2\psi(x) - \int_{-\infty}^{x} \frac{d}{dy}(y^2\psi(y))\, dy + 2\int_{-\infty}^{x} y\psi(y)\, dy$$

$$= 2\hat{T}_6 \psi(x)$$

So $[\hat{T}_2, \hat{T}_6] = 2\hat{T}_6$

$$[\hat{T}_1, \hat{T}_2]\psi(x) = x^3\left(x\frac{d}{dx}\right)\psi(x) - \left(x\frac{d}{dx}\right)x^3\psi(x)$$

$$= x^4\frac{d\psi(x)}{dx} - x^4\frac{d\psi(x)}{dx} - 3x^3\psi(x)$$

$$= -3\hat{T}_1\psi(x)$$

So $[\hat{T}_1, \hat{T}_2] = -3\hat{T}_1$

Scherrer Chapter 3 #1

2

a) $\psi(x,t) = A\, x\, e^{-\sqrt{km}/2\hbar\, x^2}\, e^{-i\sqrt{k/m}\,\frac{3}{2}t}$

Let $a = \dfrac{\sqrt{km}}{2\hbar}$ and $b = \frac{3}{2}\sqrt{k/m}$

Then $i\hbar \dfrac{\partial \psi}{\partial t} = \dfrac{-\hbar^2}{2m}\dfrac{\partial^2 \psi}{\partial x^2} + V(x)\,\psi$

$$i\hbar(-ib)\,\psi(x,t) = \frac{-\hbar^2}{2m}\frac{\partial}{\partial x}\left[-2ax\,\psi(x,t) + \psi(x,t)\right] + V(x)\,\psi(x,t)$$

$$b\hbar\,\psi(x,t) = \frac{-\hbar^2}{2m}\left[-2a\,\psi(x,t) - 2ax\left(-2ax\,\psi(x,t) + \psi(x,t)\right)\right] + V(x)\,\psi(x,t)$$

$$b\hbar\,\psi(x,t) = \frac{-\hbar^2}{2m}\left[-2a - 2ax + 4a^2x^2\right]\psi(x,t) + V(x)\,\psi(x,t)$$

$$V(x) = b\hbar + \frac{\hbar^2}{2m}\left[-2a - 2ax + 4a^2x^2\right]$$

and clearly $V(x)$ is independent of t

2 cont'd

$$\int_0^\infty x^n e^{-ax^2} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{(n+1)/2}}$$

b) $$A^2 \int_{-\infty}^{\infty} x^2 e^{-2ax^2}\, dx = 1$$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

$$A^2 \frac{\Gamma(\frac{3}{2})}{(2a)^{3/2}} = 1$$

$$A^2 = (2a)^{3/2} \frac{2}{\sqrt{\pi}}$$

$$A = (2a)^{3/4} \left(\frac{4}{\pi}\right)^{1/4}$$

$$= \left[\frac{32}{\pi}\left(\frac{\sqrt{km}}{2\hbar}\right)^3\right]^{1/4}$$

$$= \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)\right]^{1/4}$$

c) $$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \psi^* x \psi\, dx$$

$$= A^2 \int_{-\infty}^{\infty} x^3 e^{-2ax^2}\, dx$$

$$= 0 \quad \text{(since odd function of } x\text{)}$$

$$\langle p \rangle = A^2 \int_{-\infty}^{\infty} x e^{-ax^2}\left(-i\hbar \frac{\partial}{\partial x}\right)\left(x e^{-ax^2}\right) dx$$

$$= 0 \quad \text{(odd function of } x\text{)}$$

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2 c) $\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^4 e^{-2ax^2} dx$

$$= \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \frac{\Gamma(5/2)}{(2a)^{5/2}}$$

$$= \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \frac{24\sqrt{\pi}}{32} \left[\frac{\hbar}{\sqrt{km}}\right]^{5/2}$$

$$= \frac{3}{2} \frac{\hbar}{\sqrt{km}}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \left(x e^{-ax^2}\right)\left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right)\left(x e^{-ax^2}\right) dx$$

$$= \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \int_{-\infty}^{\infty} \left(x e^{-ax^2}\right)\left(-\hbar^2\right) \frac{\partial}{\partial x}\left(e^{-ax^2} - 2ax^2 e^{-ax^2}\right) dx$$

$$= \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \int_{-\infty}^{\infty} \left(x e^{-ax^2}\right)\left(-\hbar^2\right)\left(-2ax e^{-ax^2} + 4a^2x^3 e^{-ax^2} - 4ax e^{-ax^2}\right) dx$$

$$= -\hbar^2 \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \int_{-\infty}^{\infty} -6ax^2 e^{-2ax^2} + 4a^2x^4 e^{-2ax^2} \; dx$$

$$= -\hbar^2 \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \left[-6a\left(\frac{\Gamma(3/2)}{(2a)^{3/2}}\right) + 4a^2 \left(\frac{\Gamma(5/2)}{(2a)^{5/2}}\right)\right]$$

$$= -\hbar^2 \left[\frac{4}{\pi}\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \left[(-6)\frac{\sqrt{\pi}}{2}\frac{1}{2\sqrt{2a}} + (4)\left(\frac{24\sqrt{\pi}}{32}\right)\frac{1}{4\sqrt{2a}}\right]$$

$$= -\hbar^2 \left[\left(\frac{\sqrt{km}}{\hbar}\right)^3\right]^{1/2} \left(-\frac{3}{2\sqrt{2a}}\right)$$

$$= \frac{3}{2\sqrt{2}} \hbar \sqrt{km}$$

3 Scherrer Chapter 4 #7

a) $V(x) = \begin{cases} 0 & -\frac{a}{2} < x < \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$

$-\frac{\hbar^2}{2m} \psi''(x) - E\psi(x) = 0$ define $k \equiv \frac{\sqrt{2mE}}{\hbar}$

$\psi'' + k^2\psi = 0$

so $\psi(x) = A\sin(kx) + B\cos(kx)$

Boundary Conditions

$\psi(-\frac{a}{2}) = 0$ $A\sin\left(-\frac{ka}{2}\right) + B\cos\left(\frac{-ka}{2}\right) = 0$ ①

$\psi(\frac{a}{2}) = 0$ $A\sin(\frac{ka}{2}) + B\cos(\frac{ka}{2}) = 0$ ②

① + ② $\cos\left(\frac{ka}{2}\right) = 0$ true for $\frac{ka}{2} = (2n+1)\frac{\pi}{2}$ $k = \frac{(2n+1)\pi}{a}$ $n \in \mathbb{Z}$

② − ① $\sin\left(\frac{ka}{2}\right) = 0$ true for $\frac{ka}{2} = (2n)\pi$ $k = \frac{2n\pi}{a}$ $n \in \mathbb{Z}$

Let $l = 2n$

Solutions alternate between cos. and sin functions with

$$k = \frac{l\pi}{a} \quad \text{so} \quad E = \frac{\hbar^2 l^2 \pi^2}{2ma^2} \qquad l \in \mathbb{Z}$$

and $\psi(x) = \begin{cases} A\cos\left(\frac{l\pi x}{a}\right) & l \text{ odd integer} \\ A\sin\left(\frac{l\pi x}{a}\right) & l \text{ even integer} \end{cases}$

$A^2 \int_{-a/2}^{a/2} \cos^2\left(\frac{l\pi x}{a}\right) dx = 1$

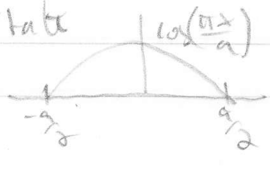
$A^2 \frac{a}{2} = 1$

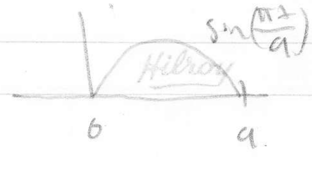
$\boxed{A = \sqrt{\frac{2}{a}}}$

same for sin case

b) The coordinate system is shifted compared to the 0 to a case but nothing else is changed so the physics is unchanged and hence the energy levels and the shape of the wave funcs.

e.g. ground state





4. $\psi(x,0) = A\sin^3\left(\frac{\pi x}{a}\right)$

$= \frac{1}{4}A\left[3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\right]$

$= \sqrt{\frac{a}{2}}\frac{1}{4}A\left[3\psi_1 - \psi_3\right]$

Set $\langle\psi|\psi\rangle = 1$

$\frac{a}{2}\frac{1}{16}A^2\left[9\langle\psi_1|\psi_1\rangle + \langle\psi_3|\psi_3\rangle - 6\langle\psi_1|\psi_3\rangle\right] = 1$ recall $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$A^2 = \frac{32}{10a}$

$A = \sqrt{\frac{32}{10a}}$

$\boxed{= \frac{4}{\sqrt{5a}}}$

b) $\psi(x,t) = \frac{1}{\sqrt{10}}\left[3\psi_1 e^{-iE_1t/\hbar} - \psi_3 e^{-iE_3t/\hbar}\right]$

$= \frac{1}{\sqrt{10}}\left[3\sin\frac{\pi x}{a} e^{-\frac{i\pi^2\hbar t}{2ma^2}} - \sin\frac{3\pi x}{a} e^{-\frac{i9\pi^2\hbar t}{2ma^2}}\right]$

$|\psi(x,t)|^2 = \frac{1}{10}\left[9\sin^2\frac{\pi x}{a} + \sin^2\frac{3\pi x}{a} - 3\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\left(e^{\frac{i4\pi^2\hbar t}{ma^2}} + e^{-\frac{i4\pi^2\hbar t}{ma^2}}\right)\right]$

$= \frac{1}{10}\left[9\sin^2\frac{\pi x}{a} + \sin^2\frac{3\pi x}{a} - 6\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\cos\frac{4\pi^2\hbar t}{ma^2}\right]$

c) $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$ with probability $|c_1|^2 = \frac{9}{10}$ or 90%

$E_3 = \frac{9\pi^2\hbar^2}{2ma^2}$ with probability $|c_3|^2 = \frac{1}{10}$ or 10%