Assignment #1

1. $E_i = 1$ MeV $E_f = 0.98$ MeV

$\theta_\gamma = \pi$

$\lambda = \frac{h}{p} = \frac{ch}{E}$

$$\lambda_f - \lambda_i = \frac{h}{m_0 c}(1 - \cos\theta)$$

Use units in which $c = 1$
so that m_0 is in MeV

$$\frac{h}{E_f} - \frac{h}{E_i} = \frac{2h}{m_0}$$

$$m_0 = \frac{2E_iE_f}{E_i - E_f}$$

$$\boxed{m_0 = 49 \text{ MeV}}$$

2 $\hat{O} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ Solve $\det(\hat{O} - \lambda \hat{I}) = 0$

So $\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$

$$-\lambda(\lambda^2 - 1) - (+\lambda) = 0$$
$$\lambda(\lambda^2 - 2) = 0$$
$$\lambda = -\sqrt{2}, 0, \sqrt{2}$$

$\lambda = -\sqrt{2}$ Solve $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\sqrt{2} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

① $b = -\sqrt{2}\, a$ ② $a + c = -\sqrt{2}\, b$ ③ $b = -\sqrt{2}\, c$

$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} b \\ b \\ -\frac{1}{\sqrt{2}} b \end{bmatrix} \Rightarrow$ normalize $\begin{bmatrix} -\frac{1}{\sqrt{2}} b & b & -\frac{1}{\sqrt{2}} b \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} b \\ b \\ -\frac{1}{\sqrt{2}} b \end{bmatrix} = 1$

$b = \frac{1}{\sqrt{2}}$

for $\lambda = -\sqrt{2}$ $\quad \frac{1}{2} \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}$

for $\lambda = 0$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$b = 0$
$a + c = 0$
$b = 0$

$\therefore \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} \Rightarrow$ normalize $\begin{bmatrix} a & 0 & -a \end{bmatrix} \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} = 1$ $\quad a = \frac{1}{\sqrt{2}}$

for $\lambda = 0$ $\quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

2. $\lambda = \sqrt{2}$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \sqrt{2} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$b = \sqrt{2}a$
$a + c = \sqrt{2}b$
$b = \sqrt{2}c$

$\therefore \frac{1}{\sqrt{2}} \begin{bmatrix} b \\ b/\sqrt{2} \\ b \end{bmatrix} \Rightarrow$ normalize $b = \frac{1}{\sqrt{2}}$

for $\lambda = \sqrt{2}$ $\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

We see that $\frac{1}{4}\begin{bmatrix} -1 & \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = 0$ and $\frac{1}{2\sqrt{2}}\begin{bmatrix} -1 & \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$

and $\frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$ $\therefore$ these three vectors are mutually orthogonal. Since it is a 3-D vector space, they form an orthonormal basis set.

$\frac{1}{2}\begin{bmatrix} -1 & \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = \frac{1}{2}(-1 + \sqrt{2}i + 1) = \frac{\sqrt{2}}{2} i$

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(2)$

$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = \frac{1}{2}(\sqrt{2}i)$

$\therefore \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} = \frac{\sqrt{2}i}{2}\left(\frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}\right) + \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) + \frac{\sqrt{2}}{2} i \left(\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}\right)$

3 a) Show $(e^z)^* = e^{z^*}$ let $z = a + ib$

$$\left(e^{a+ib}\right)^* = \left(e^a\left[\cos b + i\sin b\right]\right)^*$$
$$= e^a\left[\cos b - i\sin b\right]$$
$$= e^a e^{-ib}$$
$$= e^{a-ib}$$
$$= e^{z^*}$$

b)
$$e^{z+i\pi} = e^z\left[\cos\pi + i\sin\pi\right]$$
$$= -e^z$$

c)
$$e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$
$$= 2\cos\theta$$
$$\therefore \cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$$

$$e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$$
$$= 2i\sin\theta$$
$$\therefore \sin\theta = \frac{1}{2i}\left[e^{i\theta} - e^{-i\theta}\right]$$

d) Let $z = a + ib$

$$\sin^2(a+ib) + \cos^2(a+ib)$$
$$= (\sin a\cos ib + \cos a\sin ib)^2 + (\cos a\cos ib - \sin a\sin ib)^2$$
$$= \sin^2 a\cos^2 ib + \cos^2 a\sin^2 ib + 2\sin a\cos a\sin ib\cos ib$$
$$+ \cos^2 a\cos^2 ib + \sin^2 a\sin^2 ib - 2\sin a\cos a\sin ib\cos ib$$
$$= \cos^2 ib\,(\sin^2 a + \cos^2 a) + \sin^2 ib\,(\sin^2 a + \cos^2 a)$$
$$= \left[\frac{1}{2}\left(e^{i^2 b} + e^{-i^2 b}\right)\right]^2 + \left[\frac{1}{2i}\left(e^{i^2 b} - e^{-i^2 b}\right)\right]^2$$

$$= \frac{1}{4}\left(e^{-2b} + e^{2b} + 2\right) + \left(-\frac{1}{4}\right)\left(e^{-2b} + e^{2b} - 2\right)$$

$$= 1 \quad \text{Q.E.D.}$$

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Vectors $\{1, x, x^2, x^3, x^4\}$
Scalars $\{\mathbb{R}\}$

vector space: suppose

$$|u\rangle = a + bx + cx^2 + dx^3 + ex^4$$
$$|v\rangle = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4$$

} i.e. arbitrary vectors

① then $|u\rangle + |v\rangle = (a+\alpha) + (b+\beta)x + (c+\gamma)x^2 + (d+\delta)x^3 + (e+\varepsilon)x^4$

and clearly $|w\rangle = |u\rangle + |v\rangle$ is in the vector space for any real number $(a, b, c, d, e, \alpha, \beta, \gamma, \delta, \epsilon)$

similarly $A|u\rangle = (Aa) + (Ab)x + (Ac)x^2 + (Ad)x^3 + (Ae)x^4$ is also in the vector space for any real number A

② Similarly show $|u\rangle + |v\rangle = |v\rangle + |u\rangle$

$$(|u\rangle + |v\rangle) + |w\rangle = |u\rangle + (|v\rangle + |w\rangle)$$

③ show $A(|u\rangle + |v\rangle) = A|u\rangle + A|v\rangle$

and $(AB)|u\rangle = A(B|u\rangle)$ for any real B

④ the null vector is $|0\rangle = 0$ and $|u\rangle + |0\rangle = a + bx + cx^2 + dx^3 + ex^4 + 0$
$= |u\rangle$

⑤ the inverse of $|u\rangle$ is $|-u\rangle = -a - bx - cx^2 - dx^3 - ex^4$

and clearly $|u\rangle + |-u\rangle = 0$

⑥ $0|u\rangle = (0)a + (0)bx + (0)cx^2 + (0)dx^3 + (0)ex^4$
$= 0$
$= |0\rangle$

$1|u\rangle = (1)a + (1)bx + (1)cx^2 + (1)dx^3 + (1)ex^4$
$= |u\rangle$

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$$\psi(x) = \begin{cases} 0 & x < 0 \\ Ax(a-x) & 0 \le x \le a \\ 0 & x > a \end{cases}$$

a) $\int_0^a A^2 x^2 (a-x)^2 \, dx = 1$

$A^2 \left[\frac{a^2x^3}{3} - \frac{2ax^4}{4} + \frac{x^5}{5} \right]_0^a = 1$

$\boxed{A = \sqrt{\frac{30}{a^5}}}$

b) $\langle \hat{x} \rangle = \frac{30}{a^5} \int_0^a [x(a-x)] \, x \, [x(a-x)] \, dx$

$= \frac{30}{a^5} \left[\frac{a^2x^4}{4} - \frac{2ax^5}{5} + \frac{x^6}{6} \right]_0^a$

$\boxed{= \frac{a}{2}}$

$\langle \hat{x}^2 \rangle = \frac{30}{a^5} \int_0^a [x(a-x)] \, x^2 \, [x(a-x)] \, dx$

$= \frac{30}{a^5} \left[\frac{a^2x^5}{5} - \frac{2ax^6}{6} + \frac{x^7}{7} \right]_0^a$

$\boxed{= \frac{2}{7} a^2}$

$\sigma_{\hat{x}} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = \boxed{\frac{a}{\sqrt{28}}}$

$\langle \hat{p} \rangle = \frac{30}{a^5} \int_0^a [x(a-x)] \left(-i\hbar \frac{\partial}{\partial x} \right) [x(a-x)] \, dx$

$= \frac{-i\hbar 30}{a^5} \left[\frac{x^2a^2}{2} - \frac{3ax^3}{3} + \frac{2x^4}{4} \right]_0^a$

$\boxed{= 0}$

$\langle \hat{p}^2 \rangle = \frac{30}{a^5} \int_0^a [x(a-x)] \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) [x(a-x)] \, dx$

$= \frac{30}{a^5} (-\hbar^2)(-2) \left[\frac{ax^2}{2} - \frac{x^3}{3} \right]_0^a$

$\boxed{= \frac{10\hbar^2}{a^2}}$

$\sigma_{\hat{p}} = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = \sqrt{10} \frac{\hbar}{a}$

$\sigma_{\hat{x}} \sigma_{\hat{p}} = \sqrt{\frac{10}{28}} \hbar$

$= \sqrt{\frac{10}{7}} \frac{\hbar}{2}$

$> \frac{\hbar}{2}$