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ELG 5119 STOCHASTIC PROCESSES Fall 2014

ASSIGNMENT 3

due Tuesday, Oct. 21, 2014

1. A certain integrated circuit (IC) manufacturer produces chips with a certain cost per chip to cover the costs of producing good chips (the manufacturer can only ship chips with no defects, the rest being discarded—the cost of manufacturing a good chip must account for the manufacturing costs of the chips that are discarded). The feature size on the chip is 0.090 microns, and 90% of the chips that are tested are found to be good. The manufacturer has improved the technology so that the size of the chips can be scaled down to correspond to feature sizes of 0.060 microns, but the cost per unit area of the new process is 30% more than before; the defect rate however is 10% higher than before per unit area, still resulting in a higher yield with the new process due to the significant reduction in the area of chips. By what factor are the good chips produced by the new process going to be more or less expensive to manufacture than before? What percentage of the new chips that would be manufactured with the new process would test good [i.e., what is the *yield* of the new process]?
Remark: For those with no knowledge of IC terminology, the feature size refers to the typical dimension (length) of the smallest feature on an IC. Doubling the feature size obviously quadruples the area of an IC.

2. The performance [probability of error] of a certain digital communication system is found to be given by
$$P_e = 2Q(\sqrt{\frac{2E_b}{N_0}}) - Q^2(\sqrt{\frac{2E_b}{N_0}}).$$
Find using Table 3.1 in the notes, the value of E_b/N_0 required for P_e to be 1.2×10^{-10}; be as accurate as reasonably possible. Find P_e as accurately as possibe when $E_b/N_0 = 75$ using bounds on $Q(x)$ derived in the notes.

3. Show that if the failure rate of components in a large system increases linearly with time (i.e., $\beta(t) = \lambda t u(t)$), then the distribution of the lifetimes of components must have a Rayleigh distribution.

4. If $\mathbf{X}$ is a random variable that is uniformly distributed from -1 to 2, find the density function of the random variable $\mathbf{Y}$ defined by
$$\mathbf{Y} = \begin{cases} -\ln(1+\mathbf{X}^3), & \text{if } \mathbf{X} > 0; \\ 0, & \text{otherwise.} \end{cases}$$

5. If $\mathbf{X}$ is a random variable that is uniformly distributed from 0 to 1, find the density function of the random variable $\mathbf{Y}$ defined by
$$\mathbf{Y} = \begin{cases} \tan(\pi\mathbf{X}), & \text{if } \mathbf{X} \neq \frac{1}{2}; \\ 0, & \text{if } \mathbf{X} = \frac{1}{2}. \end{cases}$$

6. If $\mathbf{X}$ is a Gaussian random variable with standard distribution, what is the density function of the random variable $\mathbf{Y} = \mathbf{X}^2$. What type of distribution is this [from amongst the distributions catalogued].

. . . *(over)*

7. If $\mathbf{X}$ is a random variable that is uniformly distributed from 0 to 4π, find the density function of the random variable $\mathbf{Y}$ defined by

$$\mathbf{Y} = \sin(\mathbf{X} + 1).$$

Find $\mathcal{E}\{\mathbf{Y}\}$ both by applying the definition of expectation per (4.76) in the notes and by using Theorem 4.2 from the notes.

8. Rewrite the Lebesgue-Stieltjes integral

$$\int_{-\pi}^{\pi} e^{x^2}\, d\{\cos(x)u(x)\}$$

in the form

$$\int_a^b g(x)\, dx$$

for some suitable function $g(x)$ and values a and b to be specified by you.

1) In this type of question, Poisson distribution is commonly preferred;

$$P(X=k) = \lim_{n \to \infty} P(X_n = k) = \begin{cases} \frac{\mu^k}{k!} \cdot e^{-\mu} & , k=0,1,2,\dots \\ 0 & , \text{else} \end{cases}$$

Probability of a chip to be good is $P(X=0)$

$$P(X=0) = e^{-\mu} = \%90 = 0,9$$

$$e^{-\mu} = 0,9 \Rightarrow \ln(e^{-\mu}) = \ln(0,9)$$

$$\boxed{\mu = -\ln(0,9)}$$

If the size of chips are scaled down;

$$\text{area shrinks} = \frac{\left(\frac{\text{Width}}{\text{Lenth}}\right)_2^2}{\left(\frac{\text{Width}}{\text{Lenth}}\right)_1^2} = \frac{(0,06)^2}{(0,09)^2} = \frac{4}{9}$$

$$0,9 - 0,1 = 0,8 \quad (\%10 \text{ higher defect rate})$$

$$\eta = -\frac{4}{9} \cdot \ln(0,8)$$

* Yield of the new process can be found $P(X=0) = e^{-\eta}$

$$\boxed{e^{-\eta} = (0,8)^{4/9} \cong \underline{0,9056}}$$

If the production cost per chip is x, then a chip must be sold at $x/0,9$ to cover production errors.

For new production; a chip costs $x \cdot \frac{4}{9} \cdot \left(1 + \frac{30}{100}\right) = 0.58x$

then a chip must be $\frac{0,58x}{(0,8)^{4/9}} = \frac{0,58}{0,9056} \cdot x = 0,640x$

$$\boxed{\text{Factor} = \frac{0,640x}{x} = \%64 \text{ of the cost of old ones}}$$

2)

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad 1{,}2 \cdot 10^{-10}$$

3)

$$x = \sqrt{\frac{2E_b}{N_0}}$$

$$P_e = 2x - x^2 = 1{,}2 \cdot 10^{-10}$$

$$P_e = x(2-x) = 1{,}2 \cdot 10^{-10}$$

$$x^2 - 2x + 1{,}2 \cdot 10^{-10} = 0$$

$$\Delta = 4 - 4{,}8 \cdot 10^{-10}$$

$$x_{1,2} = \frac{2 \pm \sqrt{4 - 4{,}8 \cdot 10^{-10}}}{2}$$

$$x_{1,2} = 1 \pm \sqrt{1 - 1{,}2 \cdot 10^{-10}}$$

$$1 \pm \sqrt{0{,}99999999988}$$

$$\sqrt{9999999998{,}8 \cdot 10^{-10}}$$

$$1 \pm 99999 \cdot 10^{-5}$$

$$x_1 = 1{,}99999$$

$$x_2 = 0{,}00001$$

$$x = 2 \rightarrow Q(x) = 0{,}02275$$

$$\frac{2E_b}{N_0} = 4 \qquad \left|\ \frac{E_b}{N_0} = 2 \ \right|$$

for $x >> 1$;

$$Q(x) = Q\left(\sqrt{2 \cdot \frac{E_b}{N_0}}\right) = Q\left(\sqrt{150}\right) = Q(12{,}25)$$

$$x = 12{,}25$$

$$Q(x) = \frac{1}{\pi \cdot x\sqrt{2}} \cdot e^{-x^2/2}$$

$$Q(12{,}25) = 0{,}018378 \cdot 2{,}678637 \cdot 10^{-33}$$

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 4{,}923 \times 10^{-35}$$

$$\boxed{P_e = 2 \cdot Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \underbrace{Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)}_{\text{negligibly small}} = 9{,}846 \times 10^{-35}}$$

fx

if

$\sum_i f(x_i).[u(x_{i+1})-u(x_i)]$

3)

$$f_X(x) = \begin{cases} \beta(x).e^{\left(-\int_{-\infty}^{x}\beta(t).dt\right)} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

if $\beta(t) = \lambda t . u(t)$;

$$-\int_{-\infty}^{x}\beta(t) = -\left(\frac{\lambda t^2}{2} . \Big|_{-\infty}^{x}\right) = -\frac{\lambda x^2}{2}$$

$\beta(x) = \lambda . x . u(x)$

$$f_X(x) = \begin{cases} \lambda . x . e^{-\frac{\lambda x^2}{2}} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

* As can be seen this is a density function for Rayleigh Distribution.

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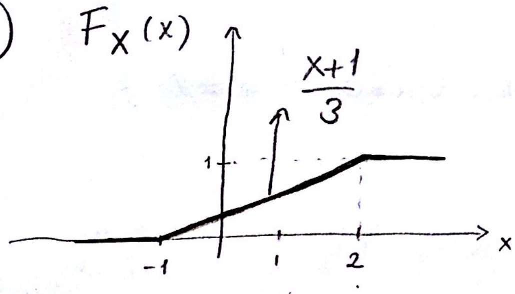
4)



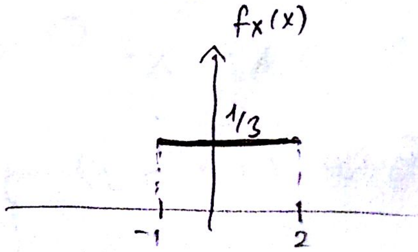
$F_X(x) = y = ax+b$

$2a+b=1$

$+a \mp b = 0$

$3a = 1$

$a = \frac{1}{3} \quad b = \frac{1}{3}$



$Y = g(X) \Rightarrow F_Y(y) = P(g(X) \leq y)$

$$Y = \begin{cases} -\ln(1+X^3) & , \quad X > 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

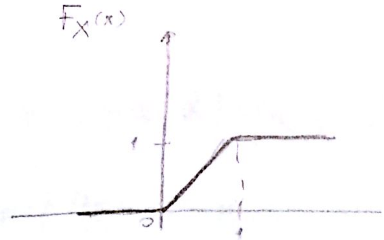
$$g'(x) = -\frac{3x^2}{1+x^3}$$

$x_i = (e^{-y}-1)^{1/3}$

$$\boxed{f_Y(y) = \begin{cases} \frac{e^{-y}}{9.(e^{-y}-1)^{2/3}} & , \quad 0 \geq y \geq -\ln 9 \\ 0 & , \quad \text{otherwise} \end{cases}}$$

$$f_Y(y) = \frac{1}{|g'(x_i)|} f_X(x_i) \rightarrow f_Y(y) = \frac{e^{-y}}{3(e^{-y}-1)^{2/3}} \cdot \underbrace{f_X(\sqrt[3]{e^{-y}-1})}_{\frac{1}{3} \text{ for } 0 \geq y \geq -\ln 9}$$

5)



$Y = \tan \pi X$

$X = \frac{1}{\pi} \tan^{-1}(Y) + a$

$F_Y(y) = P(\tan \pi X \leq y)$

$\frac{dY}{dx} = \frac{\pi}{\cos^2 \pi X} = \pi(1+\tan^2 \pi X)$

$F_Y(y) = P(X \leq \frac{1}{\pi} \tan^{-1}(y) + a)$

$\frac{dY}{dx} = \pi(1+Y^2)$

$F_Y(y) = F_X(\frac{1}{\pi} \tan^{-1}(y) + a)$

$$f_Y(y) = \frac{1}{\pi(1+y^2)} \cdot \underbrace{f_X(x)}_{1}$$

$$\boxed{f_Y(y) = \frac{1}{\pi(1+y^2)}}$$

7)

6) In standard form of Gaussian Distribution $\mu = 0$ $\sigma = 1$;

$$f_N(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

$$Y = g(X) = X^2$$

$$F_Y(y) = P(X^2 \leq y) = \begin{cases} P(-\sqrt{y} \leq X \leq \sqrt{y}) & , y \geq 0 \\ 0 & , y < 0 \end{cases}$$

$$= \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) + P(X = -\sqrt{y}) & , y \geq 0 \\ 0 & , y < 0 \end{cases}$$

$$f_Y(y) = P(X=0) \cdot \delta(y) + \begin{cases} \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})] & , y > 0 \\ 0 & , y < 0 \end{cases}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[\frac{1}{\sqrt{2\pi}} \cdot e^{-y/2} + \frac{1}{\sqrt{2\pi}} \cdot e^{-y/2}\right] = \begin{cases} \frac{1}{\sqrt{2\pi y}} \cdot e^{-y/2} & , y > 0 \\ 0 & , y < 0 \end{cases}$$

** It is "Chi-Squared distribution with n = 1"

7)

$$g(X) = Y = \sin(X+1)$$

$$X+1 = \arcsin Y + 2\pi n \qquad X+1 = 2\pi n + \pi - \arcsin Y$$

$$g'(X) = \cos(X+1)$$

$$X_1 = \arcsin Y + 2\pi n - 1$$

$$|g(X)| = \sqrt{1-y^2} \qquad X_2 = 2\pi n + \pi - \arcsin Y - 1$$

$$f_Y(y) = \frac{1}{\sqrt{1-y^2}} \cdot \sum_n \underbrace{\underbrace{f_X(\arcsin Y + 2\pi n - 1)}_{2 \cdot \frac{1}{4\pi}} + \underbrace{f_X(2\pi n + \pi - 1 - \arcsin Y)}_{2 \cdot \frac{1}{4\pi}}}_{\frac{1}{\pi}}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}}, & |y| < 1 \\ 0, & |y| > 1 \end{cases}$$

$$\boxed{\mathcal{E}\{Y\} = \int_{-\infty}^{\infty} y f_Y(y) \cdot dy = \int_{-1}^{1} \underbrace{\frac{y}{\pi\sqrt{1-y^2}}}_{\text{odd function}} \cdot dy = 0}$$

By Theorem

$$\mathcal{E}\{Y\} = \mathcal{E}\{g(X)\} = \int_0^{4\pi} \sin(X+1) \cdot \frac{1}{4\pi} \cdot dX = -\frac{\cos(X+1)}{4\pi}\Bigg|_0^{4\pi}$$

$$\boxed{\mathcal{E}\{Y\} = \frac{-\cos(1+4\pi) + \cos(1)}{4\pi} = \frac{-\cos(1+2\pi n) + \cos(1)}{4\pi} = 0} \quad n=2$$

8)

$$\int_{-\pi}^{\pi} e^{x^2} . d\{\cos(x) . u(x)\} = \int_{0}^{\pi} e^{x^2}(\delta(x) . dx - \sin(x) . dx)$$

$d\cos(x) = -\sin(x)dx$

$du(x) = \delta(x) . dx$

$$\int_{a}^{b} g(x) . dx = \int_{0}^{\pi} (e^{x^2} \delta(x) - e^{x^2} . \sin(x)) . dx$$

$$\boxed{g(x) = e^{x^2} . \delta(x) - e^{x^2} . \sin(x)}$$

$\boxed{a = 0 \quad \text{and} \quad b = \pi}$