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ELG 5119 STOCHASTIC PROCESSES Fall 2014

ASSIGNMENT 2

due Tuesday, October 7, 2014

1. Find the probability that when three dice are thrown, the sum of the numbers that turn up is larger than 12.

2. Eight rooks (or "castles") are placed on a chess board "at random". What is the probability that the placement is such that no rook can take another?
Hint: No rooks can take each other when each column and row on the chess board contains exactly one rook.

3. In the problem of N couples at the party where everyone chooses a partner for a dance, show that the probability that exactly k men end up dancing with their wives is exactly

$$P_N(k) = \frac{1}{k!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right\}.$$

What is the limit of this as N becomes arbitrarily large?

4. In Example 2.9 in the notes, suppose the professor only distributes 50 copies to students the first day, leaving the professor 10 copies to start with the next day. What is the probability that there are no problems the next day in this scenario? Is there an optimum number of reports to distribute the next day if we seek to minimize the probability of a problem occuring. If so, what is that number?

5. Give an example of a probability space and a real-valued function defined on the sample that is not a random variable, and another real-valued function defined on the same sample space that is a random variable.

6. Suppose that for some random variable $\mathbf{X}$ we are given the functions
(i) $G_1(x) \triangleq \mathcal{P}(x < \mathbf{X} \leq x+1)$,
(ii) $G_2(x) \triangleq \mathcal{P}(x-1 < \mathbf{X} < x)$,
(iii) $G_3(x) \triangleq \mathcal{P}(\mathbf{X}^2 \leq x)$,
(iv) $G_4(x) \triangleq \mathcal{P}(\tan^{-1}(\mathbf{X}) > x)$.
Which of these functions, if any, on its own constitutes a complete probabilistic description of $\mathbf{X}$. For any that are, find $\mathcal{P}(\mathbf{X} \leq 0)$ in terms of that $G_i(x)$. For any that are not, explain why not.

1. Let's assume first dice turns up 6;

6	16,	25,	34,	43,	52,	61
5		26	35	44	53	62
4			36	45	54	63
3				46	55	64
2					56	65
1						66

Larger than 12;

For three dices total probability is 6^3

* There is 1 possibility when first dice is 1

* There are 3 possibilities when " " " 2

* " " 6 " " " " " 3

* " " 10 " " " " " 4

* " " 15 " " " " " 5

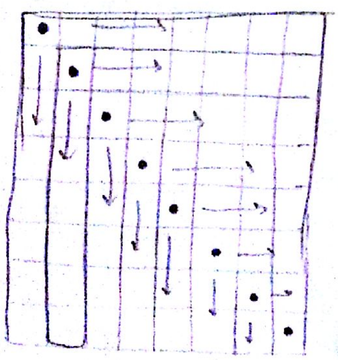
* " " 21 " " " " " 6

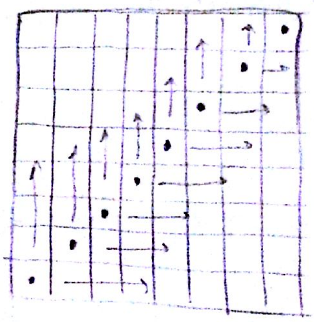
+

Total : 56

$$P\left(\begin{array}{l}\text{When three dice are thrown}\\ \text{the sum of the numbers is} > 12\end{array}\right) = \frac{56}{6^3} = \frac{56}{216} = \boxed{\frac{7}{27}}$$

2. With these two placement, no castles can take the other.





$$P\left(\begin{array}{l}\text{No rook}\\ \text{can take another}\end{array}\right) = \frac{2\cdot\binom{15}{1}\binom{13}{1}\binom{11}{1}\binom{9}{1}\binom{7}{1}\binom{5}{1}\binom{3}{1}\binom{1}{1}}{\binom{64}{8}} = \boxed{\frac{9,16}{10000}}$$

3. First we need to select these k men from N men. $\binom{N}{k}$

These selected k men will dance with their couples

Rest of the men are not allowed to dance with their own wives.

We know from the example in the book;

The probability that no one dances with their spouses, is for n couples

$$P\left(\substack{\text{no one}\\ \text{dances}\\ \text{with n}\\ \text{Spouse}}\right) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \cdots\cdots + \frac{(-1)^n}{n!}$$

But we have $(N-k)$ instead of n;

$$P\left(\substack{\text{no one}\\ \text{dances}\\ \text{with}\\ \text{spouse}\\ N-k}\right) = (N-k)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \cdots\cdots + \frac{(-1)^{N-k}}{(N-k)!}\right)$$

$$P\left(\substack{\text{k men dances}\\ \text{with their}\\ \text{wives}}\right) = \frac{\binom{N}{k}.(N-k)!}{N!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots\cdots + \frac{(-1)^{N-k}}{(N-k)!}\right)$$

$$= \frac{\cancel{N!}}{\cancel{N!}.\cancel{(N-k)!}\,k!} . \cancel{(N-k)!}$$

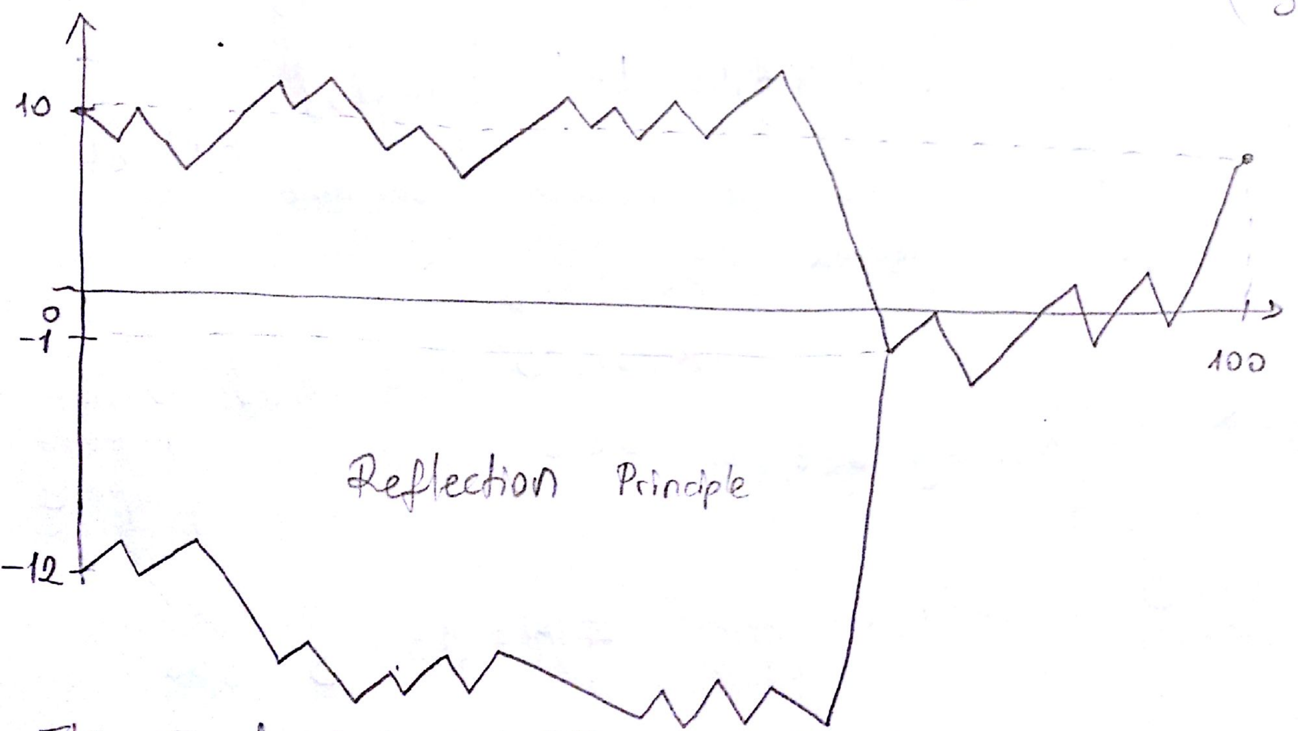
$$= \frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots\cdots + \frac{(-1)^{N-k}}{(N-k)!}\right)$$

$$e^{-1} = \frac{1}{e} = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots\cdots\right)$$

if $N \to \infty$;

$$P\left(\substack{\text{exactly}\\ \text{k men}\\ \text{dances with}\\ \text{their wives}}\right) = \frac{1}{e.k!}$$

4. In this case total # of trajectories is $\binom{100}{50}$



The # of bad trajectories is $\binom{100}{39}$ because there are 61 up 39 down movements.

The prob. that there is no problem is;

$$P(\text{no prob}) = 1 - \frac{\binom{100}{39}}{\binom{100}{50}} = 1 - \frac{(\cancel{100!}).(50!)(50!)}{(61!)(39!)(\cancel{100!})}$$

$$= 1 - \frac{50.49.48.47.46.45.44.43.42.41.40}{61.60.59.58.57.56.55.54.53.52.51} = 0,911$$

To minimize the probability of problem occurring;

$$\frac{\binom{100}{39}}{\binom{100}{40+x}} = \frac{\cancel{100!}.(40+x)!(60-x)!}{\cancel{100!}(39)!.61!} = \ldots \qquad \boxed{X = 10}$$

we need to maximize this; $\binom{100}{50} \geqslant \binom{100}{n}$

*Professor is doing a great job by distributing 50 copies in the first day because he minimizes the prob. of problem occurring next day.

5. Lets have a universal set $S = \{6,7,8,9,10,11\}$
and event space $\tilde{F} = \{S, \emptyset, \{6,7,8\}, \{9,10,11\}\}$

$\Rightarrow X(s) = s$ is not a random variable
because $X(s) \le y$
$"X(9) \le 9" = \{6,7,8,9\}$ is not in event space
$(9 \in S)$

* for all $y \in \mathbb{R}$ $\{s \mid X(s) \le y, s \in S\} \in \tilde{F}$

$\Rightarrow Z(s) = 0$ is a r.v because $"Z(s) \le y" = \begin{cases} \emptyset & y<0 \\ S & y>0 \end{cases}$

and $\emptyset$ and S are in event space $\tilde{F}$

6. i) If we can find $F_X(x)$ from $G_1(x)$, it will constitute a complete probabilistic description of X.

$G_1(x) = P(x < X \le x+1) = F_X(x+1) \overset{!}{-} F_X(x)$

$P(X \le x+1) = F_X(x+1)$

$P(X > x) = - F_X(x)$

$G_1(x) = F_X(x+1) - \cancel{F_X(x)}$

$G_1(x-1) = \cancel{F_X(x)} - \cancel{F_X(x-1)}$

$G_1(x-2) = \cancel{F_X(x-1)} - \cancel{F_X(x-2)}$

$G_1(x-3) = \cancel{F_X(x-2)} - F_X(x-3)$

$\vdots$

$G_1(x-n) = \cancel{F} - F_X(x-n)$

$G_1(x) + G_1(x-1) + \dots G_1(x-n) =$
$= F_X(x+1) - F_X(x-n)$

$$\sum_{k=0}^{n} G_1(x-k) = F_X(x+1) - F_X(x-n)$$

$$\sum_{k=0}^{\infty} G_1(x-k) = F_X(x+1) - \underbrace{\lim_{N\to\infty} F_X(x-N)}_{0}$$

$= F_X(x+1)$

$F_X(x)$ can be obtained from $G_1(x)$ so it's a complete probabilistic description of X.

$$\boxed{P(X \leq 0) = \sum_{k=0}^{\infty} G_1(-1-k)}$$

ii) $G_2(x) = P(x-1 < X < x)$;

$$P(x-1 < X < x) = P(X < x) - P(X \leq x-1)$$

$$= F_X(x) - P(X = x) - F_X(x-1)$$

$$= \cancel{F_X(x)} - \cancel{F_X(x)} + F_X(x-0) - F_X(x-1)$$

$$G_2(x) = F_X(x-0) - F_X(x-1)$$

$$G_2(x-1) = F_X(x-1) - F_X(x-2)$$

$$G_2(x-2)$$

$$\vdots$$

$$G_2(x-n) = F_X(x-n) - F_X(x-n-1)$$

$$\sum_{k=0}^{\infty} G_2(x-(2k-1)) = F_X(x) - F_X(x-n)$$

$\lim_{n\to\infty} \to 0$

Complete Prob. Description of X

$$P(X \leq 0) = \sum_{k=0}^{\infty} G_2(0-(2k-1))$$

iii) $G_3(x) = P(X^2 \leq x)$

iv) $G_4(x) = P(\tan^{-1}(X) > x) = \begin{cases} 0 & x \geq \pi/2 \\ P(X > \tan x & |x| < \pi/2 \\ 1 & x < -\pi/2 \end{cases}$

$$= \begin{cases} 0 & x \geq \pi/2 \\ 1 - F_X(\tan(x)) & |x| < \pi/2 \\ 1 & x < -\pi_2 \end{cases}$$

$1 - G_4(\tan^{-1}(x)) = F_X(x)$

Complete prob. description of X

$P(X \leq 0) = 1$

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