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ELG 5119 STOCHASTIC PROCESSES Fall 2014

ASSIGNMENT 1

due at the start of class, Thursday, Sept. 25, 2014

1. Show that (a) $\overline{\overline{\mathcal{A}}+\overline{\mathcal{B}}}+\overline{\overline{\mathcal{A}}+\mathcal{B}}=\mathcal{A}$; (b) $(\mathcal{A}+\mathcal{B})(\overline{\mathcal{A}\mathcal{B}})=\mathcal{A}\overline{\mathcal{B}}+\mathcal{B}\overline{\mathcal{A}}$.

2. Consider the sequence of subsets of the real line $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\ldots$ where $\mathcal{A}_i \triangleq [1-(1/i^2), 5+(2/i))$. Find $\bigcup_{i=1}^{\infty}\mathcal{A}_i$ and $\bigcap_{i=1}^{\infty}\mathcal{A}_i$

3. If a collection of sets $\widetilde{\mathcal{F}}$ is such that (i) $\mathcal{S}\in\widetilde{\mathcal{F}}$, and (ii) if $\mathcal{A},\ \mathcal{B}\in\widetilde{\mathcal{F}}$ then $\mathcal{A}\triangle\mathcal{B}\in\widetilde{\mathcal{F}}$, is it necessary that $\widetilde{\mathcal{F}}$ be an algebra? If so, prove it. If not, give an example of a collection of sets for which (i) and (ii) hold, but the collection is not an algebra.

4. Show that (a) if $\mathcal{P}(\mathcal{A})=\mathcal{P}(\mathcal{B})=\mathcal{P}(\mathcal{A}\mathcal{B})$, then $\mathcal{P}(\mathcal{A}\overline{\mathcal{B}}+\mathcal{B}\overline{\mathcal{A}})=0$; (b) if $\mathcal{P}(\mathcal{A})=\mathcal{P}(\mathcal{B})=1$, then $\mathcal{P}(\mathcal{A}\mathcal{B})=1$.

5. Prove and generalize the following identity:

$$\mathcal{P}(\mathcal{A}+\mathcal{B}+\mathcal{C})=\mathcal{P}(\mathcal{A})+\mathcal{P}(\mathcal{B})+\mathcal{P}(\mathcal{C})-\mathcal{P}(\mathcal{A}\mathcal{B})-\mathcal{P}(\mathcal{A}\mathcal{C})-\mathcal{P}(\mathcal{B}\mathcal{C})+\mathcal{P}(\mathcal{A}\mathcal{B}\mathcal{C}).$$

6. Show that $\mathcal{P}(\mathcal{A}\mathcal{B}\mid\mathcal{C})=\mathcal{P}(\mathcal{A}\mid\mathcal{B}\mathcal{C})\mathcal{P}(\mathcal{B}\mid\mathcal{C})$ and $\mathcal{P}(\mathcal{A}\mathcal{B}\mathcal{C})=\mathcal{P}(\mathcal{A}\mid\mathcal{B}\mathcal{C})\mathcal{P}(\mathcal{B}\mid\mathcal{C})\mathcal{P}(\mathcal{C})$. Assume that none of the conditioning events is an almost null event; what happens if one of the conditioning events is an almost null event?

7. Show that $\mathcal{A}$ and $\mathcal{B}$ are independent events if and only if $\overline{\mathcal{A}}$ and $\mathcal{B}$ are independent.

8. Suppose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are independent events. Show then that (i) $\mathcal{A}\mathcal{B}$ and $\mathcal{C}$ are two independent events, (ii) $\mathcal{A}+\mathcal{C}$ and $\mathcal{D}$ are two independent events, and (iii) $\mathcal{A}+\mathcal{B}$ and $\mathcal{C}+\mathcal{D}$ are two independent events.

9. Is it possible that an event is independent of itself? If so, when? If not, explain mathematically why not?

10. We have two coins; the first is fair and the second is unfair as it has a head on both sides. We pick one "at random" and toss it twice and a head comes up both times. What is the probability that the fair coin was picked?

11. Suppose of the set $\mathcal{S}=\{0,1,2,3\}$ we say that two elements a and b are *similar*, written $a\sim b$, when $b-a$ is either 0 or 2. Determine which elements are similar to which other, and whether the "$\sim$" relationship is an equivalence relationship on $\mathcal{S}$. If it is, prove it. If not, which of the three required properties of being reflexive, symmetric and transitive does this relationship have?

STOCHASTIC PROCESSES
ASSIGNMENT I

1. a) By using De Morgan's Rules;

$$\overline{\bar{A}+\bar{B}} = AB$$

$$\overline{\bar{A}+B} = A\bar{B}$$

$$\Longrightarrow \boxed{\overline{\bar{A}+\bar{B}} + \overline{\bar{A}+B} = A\underbrace{(B+\bar{B})}_{\text{By using boolean algebra } 1} \overset{\checkmark}{=} A}$$

b) $(A+B)(\overline{AB}) = (A+B)(\bar{A}+\bar{B})$

$$= \underbrace{A\bar{A}}_{0} + \bar{A}B + A\bar{B} + \underbrace{B\bar{B}}_{0}$$

$$\boxed{(A+B)(\overline{AB}) \overset{\checkmark}{=} \bar{A}B + A\bar{B}}$$

2.

$$A_i \triangleq \left[1-(1/i^2), 5+(2/i)\right)$$

$$A_1 = [0,7)$$

$$A_2 = [3/4, 6)$$

$$\vdots$$

$$A_\infty = (1,5)$$

$$\boxed{\bigcup_{i=1}^{\infty} A_i = [0,7) \qquad \bigcap_{i=1}^{\infty} A_i = (1,5)}$$

3. If a collection of sets $\tilde{\mathcal{F}}$ is such that;

i) $S \in \tilde{\mathcal{F}}$

ii) if $A, B \in \tilde{\mathcal{F}}$ then $A \Delta B \in \tilde{\mathcal{F}}$

* $\tilde{\mathcal{F}}$ is called an algebra if;

i) $\emptyset, S \in \tilde{\mathcal{F}}$

ii) $A, B \in \tilde{\mathcal{F}} \Rightarrow AB \in \tilde{\mathcal{F}}$

iii) $A \in \tilde{\mathcal{F}} \Rightarrow \bar{A} \in \tilde{\mathcal{F}}$

* $\tilde{\mathcal{F}} = \{ \emptyset, S, \{A\} \{B\}\}$ has to be an algebra;

* $(A-B)+(B-A) \in \tilde{\mathcal{F}}$

* $S \in \tilde{\mathcal{F}}$

if A and B are equal sets then $A \Delta B = \emptyset$ and it is in the algebra.
if A and B are disjoint $A \Delta B = A+B = S$ and it is in the algebra

$\tilde{\mathcal{F}}$ satisfies closure under symmetric difference, closure under complementation and has null set and universe as elements; therefore $\tilde{\mathcal{F}}$ is an algebra.

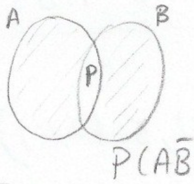
4. Show that;

(a) if $P(A) = P(B) = P(AB)$ then $P(A\bar{B} + B\bar{A}) = 0$

$P(AB) = P(A) + P(B) - P(A+B)$

$P(AB) = P(A+B)$

$P(A) = P(B) = P(A+B)$



$$P(A\bar{B} + B\bar{A}) = \underbrace{P(A+B)}_{P(AB)} - P(AB)$$

$$\boxed{P(A\bar{B}+B\bar{A}) \overset{\checkmark}{=} 0}$$

b) if $P(A) = P(B) = 1$ then $P(AB) = 1$

$$\underbrace{P(B)}_{1} = \underbrace{P(B-A)}_{0} + P(AB)$$

$$\boxed{P(AB) \overset{\checkmark}{=} 1}$$

if these are disjoint;

$$P(B-A) = \underbrace{P(B)}_{1} - \underbrace{P(A)}_{1} = 0$$

5. Prove that;

$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

* <u>Let $X = A+B$</u>;

$P(X+C) = P(X) + P(C) - P(XC)$

$P(X) = P(A+B) = P(A) + P(B) - P(AB)$

$P(XC) = P(AC) + P(BC) - P(ABC)$

$$P(A+B+C) = \underbrace{P(A+B)} + P(C) - P(AC) - P(BC) + P(ABC)$$

$$\boxed{P(A+B+C) \overset{\checkmark}{=} P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)}$$

6. show that i) $P(AB|C) = P(A|BC)\,P(B|C)$

ii) $P(ABC) = P(A|BC) \cdot P(B|C)\,P(C)$

i) $P(AB|C) = \dfrac{P(ABC)}{P(C)}$

$P(A|BC) = \dfrac{P(ABC)}{P(BC)}$ $\qquad \dfrac{P(ABC)}{P(C)} \overset{\checkmark}{=} \dfrac{P(ABC)}{P(\cancel{BC})} \cdot \dfrac{P(\cancel{BC})}{P(C)} = \dfrac{P(ABC)}{P(C)}$

$P(B|C) = \dfrac{P(BC)}{P(C)}$

ii) $P(A|BC) = \dfrac{P(ABC)}{P(BC)}$

Chain Rule;

$$P(ABC) \overset{\checkmark}{=} \frac{P(ABC)}{\cancel{P(BC)}} \cdot \frac{\cancel{P(BC)}}{\cancel{P(C)}} \cdot \cancel{P(C)}$$

$P(B|C) = \dfrac{P(BC)}{P(C)}$

If one of the conditioning events is null event; chain rule can no longer be applied.

7. if $\bar{A}$ and B are independent; (Event B is independent of the condition)

$P(B|\bar{A}) = P(B)$

Baye's rule;

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(\bar{A}) \cdot P(B|\bar{A})} = P(A)$$

*In order to A and B to be independent } $P(A|B) = P(A)$

$P(B|A) = P(B)$

$$\frac{P(A),\ P(\cancel{B})}{P(A).\ \cancel{P(B)} + P(\bar{A}).\ \cancel{P(B)}} = P(A)$$

$\dfrac{\cancel{P(A)}}{P(A) + P(\bar{A})} = \cancel{P(A)}$ $\qquad P(A) + P(\bar{A}) \overset{\checkmark}{=} 1$

This is one of the properties of probability.

8. A, B, C, D are independent events. Show that;

i) AB and C are independent.

* Baye's Rule was used to solve this question.

$$P(ABCD) = P(A).P(B).P(C).P(D)$$

$$P(AB|C) = \frac{P(AB).P(C|AB)}{P(AB).P(C|AB) + P(\overline{AB}).P(C|\overline{AB})} = ?$$

* if AB and C are independent events;

$P(C|AB) = P(C)$

$P(C|\overline{AB}) = P(C)$

$P(AB|C) = P(AB)$

$$P(AB|C) = \frac{P(AB).\cancel{P(C)}}{\underbrace{P(AB).\cancel{P(C)} + P(\overline{AB}).\cancel{P(C)}}_{1}} \overset{\checkmark}{=} P(AB)$$

independent ✓

ii)

$$P(A+C) = P(A) + P(C) - P(A).P(C)$$

$$P((A+C)|D) = \frac{(P(A)+P(C) - P(A).P(C))\cancel{P(D)}}{\underbrace{(P(A)+P(C) - P(A).P(C)).\cancel{P(D)} + (P(\bar{A}) + P(\bar{C}) - P(\bar{A}).P(\bar{C}))\cancel{P(D)}}_{2 \quad -1}} \overset{\checkmark}{=} \frac{P(A+C)}{1}$$

independent

iii) can be showed by using the same method used in ii)

9. Yes, It is possible. An event A is independent of itself if $P(A) = 1$ or $P(A) = 0$.

$$P(AA) = P(A).P(A)$$

$$P(A) = P(A).P(A)$$

$$x = x^2 \quad \text{when} \quad x = 0 \text{ or } x = 1$$

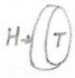
also; conditional probability;

$$P(A|A) = P(A) = \frac{P(A).P(A|A)}{P(A).P(A|A) + P(\bar{A}).P(A|\bar{A})}$$

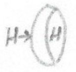
$$\cancel{P(A)} = \frac{\cancel{P(A)}.P(A)}{P(A)^2 + (1-P(A)).P(A)} = \frac{P(A)}{P(A)} \overset{\checkmark}{=} 1$$

10.

$\tilde{F}_1$



$\tilde{F}_2$



$\tilde{F}_1 = \{\{H\}, \{T\}, S, \emptyset\}$

$\tilde{F}_2 = \{\{H\}, S, \emptyset\}$

$$? = \frac{P(\text{Fair coin is selected and comes up 2 Heads})}{P(\text{Fair coin is selected and comes up 2 Heads}) + P(\text{Unfair coin is selected and 2 Heads comes up})}$$

$$P\left(\begin{matrix}\text{Fair coin to be} \\ \text{selected and} \\ \text{2 Heads comes up}\end{matrix}\right) = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 . 1} = \boxed{\frac{1}{5}}$$