MAT 1320E Calculus I
Midterm 2 Solution

Professor: Wanshun Wong

1. [2 points] Find the derivative of $f(x) = x^{\sin x}$.

We use logarithmic differentiation: $\ln f(x) = \ln(x^{\sin x}) = \sin x \ln x$. Hence by Chain Rule we have

$$\frac{f'(x)}{f(x)} = \cos x \ln x + \sin x \cdot \frac{1}{x}$$

So $f'(x) = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$.

2. [2 points] A ladder 5 m long is leaning against a wall. The top of the ladder is sliding down the wall at 0.5 m/s. How fast is the bottom of the ladder sliding along the floor when the ladder is 3 m from the wall ?

By Pythagoras theorem $x^2 + y^2 = 25$. Differentiating both sides with respect to t we get

$$2xx' + 2yy' = 0.$$

When $x = 3$, $y = 4$. We also have $y' = -0.5$ (negative because the ladder is sliding down). Therefore we get $x' = \frac{2}{3}$.

3. [1 point] What is $\displaystyle \frac{d}{dx}\left(\int_1^{x^2} \frac{3t + 5e^{2t}}{\sqrt{t^3 + \cos t}}\, dt\right)$?

By Fundamental Theorem of Calculus and Chain Rule, the derivative is $\displaystyle \frac{2x(3x^2 + 5e^{2x^2})}{\sqrt{x^6 + \cos x^2}}$

4. [5 points] Evaluate the following integrals:

(a) $\displaystyle \int \frac{2e^x}{1 + e^{2x}}\, dx$

Let $u = e^x$, so $du = e^x dx$. Then

$$\int \frac{2e^x}{1 + e^{2x}}\, dx = \int \frac{2}{1 + u^2}\, du = 2 \arctan u + C = 2 \arctan(e^x) + C.$$

(b) $\int x^5 \ln x \, dx$

Let $u = \ln x, v = \frac{x^6}{6}$, so $du = \frac{1}{x}, dv = x^5 dx$. Then

$$\int x^5 \ln x \, dx = \frac{x^6}{6} \ln x - \int \frac{x^6}{6} \cdot \frac{1}{x} \, dx = \frac{x^6 \ln x}{6} - \frac{x^6}{36} + C$$

(c) $\int \frac{1}{x^2 + 4x + 5} \, dx$

Since $\int \frac{1}{x^2 + 4x + 5} \, dx = \int \frac{1}{(x+2)^2 + 1} \, dx$, by using the subsitution $u = x + 2$ we see that the integral is $\arctan(x + 2) + C$.

(d) $\int_0^{\pi} \sin^3 x \cos^2 x \, dx$

Let $u = \cos x$, then $du = -\sin x dx$. When $x = 0, u = 1$, and when $x = \pi, u = -1$. Therefore

$$\int_0^{\pi} \sin^3 x \cos^2 x \, dx = \int_1^{-1} -(1 - u^2)u^2 \, du = (\frac{u^3}{3} - \frac{u^5}{5}) \mid_{-1}^{1} = \frac{4}{15}.$$

(e) $\int_1^{e} \frac{2 \ln x}{x} \, dx$

Let $u = \ln x$, then $du = \frac{1}{x} dx$. When $x = 1, u = 0$, and when $x = e, u = 1$.

$$\int_1^{e} \frac{2 \ln x}{x} \, dx = \int_0^{1} 2u \, du = 1.$$

5. [2 points] Find the linear approximation of $f(x) = \sqrt[3]{8 + x}$ at $x = 0$ and use it to estimate $\sqrt[3]{9}$.

We have $f'(x) = \frac{1}{3(8 + x)^{\frac{2}{3}}}$, so $f'(0) = \frac{1}{12}$. The linear approximation is

$$f(x) \approx 2 + \frac{1}{12}(x - 0) = 2 + \frac{x}{12}.$$

Then $\sqrt[3]{9} = f(1) \approx 2 + \frac{1}{12} = \frac{25}{12}$.

6. [2 points] Use R_4 to estimate the area under the curve $y = \dfrac{1}{1+x}$ between $x = 0$ and $x = 1$.

We have $\Delta x = \frac{1}{4}$, and the right endpoints of the subintervals are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. Therefore

$$R_4 = \frac{1}{4}\left(\frac{1}{1+0.25} + \frac{1}{1+0.5} + \frac{1}{1+0.75} + \frac{1}{1+1}\right) = \frac{533}{840}.$$

7. [1 point] What is the area under the curve $y = \dfrac{1}{1+x}$ between $x = 0$ and $x = 1$?

We have

$$A = \int_0^1 \frac{1}{1+x}\, dx = \ln(1+x)|_0^1 = \ln 2.$$