McGill University Math 262: Intermediate Calculus - May 2006 Midterm Exam

Best 6 out of 8 questions - you may answer all seven questions

All questions are worth 10 points each.

Potentially helpful stuff:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7183\ldots$$

Problem 1.

(a) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty}\frac{9^n}{\sqrt{n}}(x-1)^n$$

(b) Let

$$f(x) = \frac{e^{-\frac{1}{2}x^3} - 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6}{x}$$

Compute the 26th derivative of $f(x)$ at $x = 0$.

Problem 2.

Determine whether the given series converge, diverge to $\pm\infty$ or do not converge.

(a)

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$

(b)

$$\sum_{n=1}^{\infty}\frac{5^n\ln(n)}{n!}$$

Problem 3.

Let $y(x)$ be a solution of the differential equation

$$(1-x^2)y'' - y' - xy = 0, \quad y(0) = 1, y'(0) = 0.$$

Find the power series expansion of $y(x)$ about the point $x = 0$.

Problem 4.

Give explanation for your answers.

(a) Find the distance between the point $P = (1, 1, 1)$ and the plane that contains the point $(0, 0, 1)$ and has normal vector $\mathbf{n} = (-1, 2, 1)$.

(b) Find the shortest distance between two lines $x = 3t, y = t, z = -1$ and $x = 1, y = 2t, z = -t + 1$.

Problem 5.

(a) Consider the curve $\mathbf{r}(t) = (\frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t)$. Find the arclength of the segment from $t = 0$ to $t = 8$.

(b) Parametrize the curve given by the intersection of the two surfaces

$$x^2y + z^2 = 1$$
$$x - x^3z = -1$$

Problem 6.

Consider the curve $\mathbf{r}(t) = (t^3, t, -t^2)$. Find the unit tangent vector, unit normal vector, unit binormal vector, curvature and torsion of the curve at the point at t=1. That is find $\hat{T}(t), \hat{N}(t), \hat{B}(t), \kappa(t)$ and $\tau(t)$. *Note: You must do the complete calculations to obtain full points for this question (i.e. don't just leave your answers in terms of cross products and dot products of vectors).*

Problem 7.

(a) Let $f(x, y) = \sin(e^{x^2+3y} - xy)$ and $g(t) = e^{3t^2}$. Compute $f_x(x, y)$, $f_y(x, y)$, $\frac{\partial}{\partial t}\Big(f(x, g(t))\Big)$ and $\frac{\partial}{\partial t}\Big(f(t, g(t))\Big)$. You may leave your answers in terms of $g(t)$.

(b) Find the limit if it exists or show it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + y^3}{x^3 - xy^2}$$

Problem 8.

(a) Let $f(x, y) = e^{x^3y} - \frac{2x}{1+y}$. Find the equation of the tangent plane to the graph of $f(x, y)$ at the point $(x, y) = (1, 0)$

(b) Use the equation of the tangent plane obatined in part (a) to obtain an approximate value of $f(0.9, 0.1)$.