Questions 1–5 are multiple choice. Enter the **letter** corresponding to each correct answer in the appropriate box below.

Question	1	2	3	4	5	6
Answer	C	D	C	C	B	D

1. How many binary strings of length 10 start with 01 and contain at most four 0s?

   A. 12 **B.** 176 [✓ **C.** 93] **D.** 255 **E.** 32 **F.** 194
   G. None of the above.

(# bin. strings of len. 10 starting with 01 and ≤ 4 0s)

= (# bin. strings of len. 8 with ≤ 3 0s)

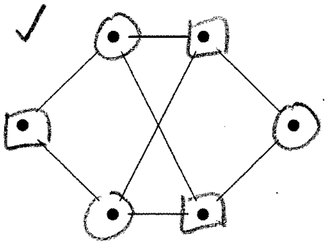
= $\sum_{i=0}^{3}$ (# bin. strings of len. 8 with exactly i 0s)

= $\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\binom{8}{3}=$

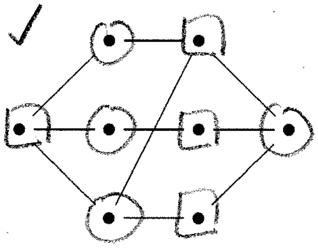
= $1+8+\frac{8\cdot7}{2}+\frac{8\cdot7\cdot6}{3\cdot2} = 9+28+56 = \underline{\underline{93}}$

[2pts] 2. Which of the following graphs are bipartite?

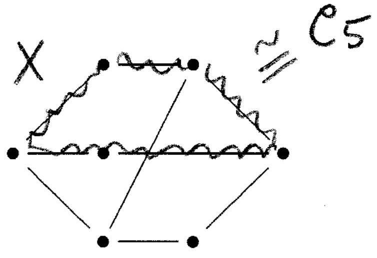
G : ✓



H : ✓



K : X



≅ C_5

A. Only *G* **B.** Only *H* **C.** Only *K* [✓ **D.** Only *G* and *K*]
E. Only *G* and *H* **F.** All of them. **G.** None of them.

3. Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 2, 4, 6, 8\}$. The number of onto (surjective) functions from A to B is:

   A. 1 **B.** 5 [✓ **C.** 5!] **D.** 2^5 **E.** 5^5 **F.** 5^2
   G. None of the above.

Since $|A| = |B|$, every onto function $f : A \to B$ is also a bijection (and hence one-to-one).

Hence (# such functions) =

$= $ (# one-to-one functions $A \to B$)

$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \underline{\underline{5!}}$

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4. Let A, B, C be arbitrary subsets of the universal set $\mathcal{U}$. Which of the following statements is necessarily true?

   A. $A \cap B \subseteq A \cap C$ implies $B \subseteq C$
   B. $A \cup B \cup C = \mathcal{U}$ implies $(A \cup B) - (A \cup C) = \emptyset$
   [✓ **C.** $A \subseteq B \cup C$ implies $A - B \subseteq C - B$.]
   D. $A \cap B \subseteq C$ implies $(C - A) \cap (C - B) = \emptyset$.
   E. None of the above.

"A" is false: e.g. $A = \{1, 2\}$, $C = \{1, 2, 3\}$, $B = \{1, 4\}$
$A \cap C = \{1, 2\}$, $A \cap B = \{1\}$ so $A \cap B \subseteq A \cap C$
but $B \not\subseteq C$

"B" is false: e.g. $\mathcal{U} = \{1, 2, 3\}$, $A = \{1\}$, $B = \{2\}$, $C = \{3\}$
$A \cup B \cup C = \mathcal{U}$, $A \cup B = \{1, 2\}$, $A \cup C = \{1, 3\}$,
$(A \cup B) - (A \cup C) = \{2\}$

"C" is true: $x \in A - B \Rightarrow (x \in A \wedge x \notin B) \Rightarrow (x \in B \cup C \wedge x \notin B)$
$\Rightarrow (x \in C \wedge x \notin B) \Rightarrow (x \in C - B)$

"D" is false: e.g. $A = B = \{1\}$, $C = \{1, 2\}$, $A \cap B = \{1\} \subseteq C$
$(C - A) \cap (C - B) = C - \{1\} = \{2\} \neq \emptyset$

5. From an urn containing balls numbered 1–10, we randomly draw three different balls, and record the sum of the three numbers. What is the smallest number of times we need to repeat this procedure to guarantee that the same sum shows at least twice?

   A. 22 [✓ **B.** 23] **C.** 27 **D.** 28 **E.** 120 **F.** 121
   G. None of the above.

Possible sums: 1+2+3, 1+2+4, .., 8+9+10

so 6, 7, .., 27; that is, 22 possible sums.

Boxes: sums 6, 7, .., 27

Objects: sums of the triples of balls drawn

By the Pigeonhole Principle, we need at least 23 draws..

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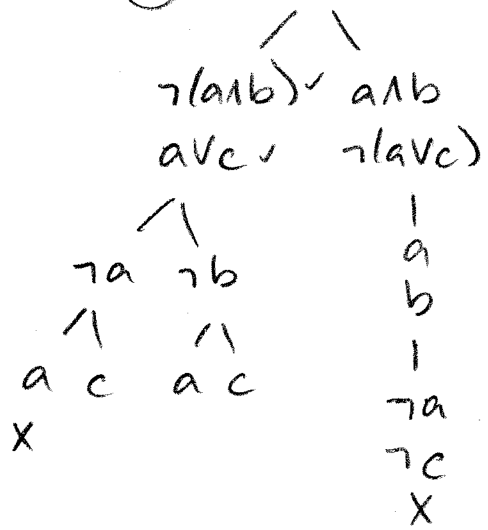
6. Which of the following is logically equivalent to $(\neg b \wedge a) \vee (\neg a \wedge c) \vee (\neg b \wedge c)$? (*)

   A. $\neg(a \wedge c) \leftrightarrow (c \vee b)$ **B.** $\neg(a \wedge \neg b) \rightarrow (\neg a \wedge \neg c)$ **C.** $(a \wedge b) \leftrightarrow (c \vee \neg a)$
   [✓ **D.**] $\neg(a \wedge b) \leftrightarrow (a \vee c)$ ✓ **E.** $\neg(c \wedge a) \rightarrow (a \wedge b)$



$\neg(a \wedge b) \Leftrightarrow a \vee c \equiv (\neg a \wedge c) \vee (a \wedge \neg b) \vee (\neg b \wedge c)$

(Convince yourself that the DNFs of all other propositions are non-equivalent to (*))

True/false questions — circle T (true) or F (false). You need not justify your answers.

7. For each of the statements below, determine whether it is true or false. *Circle each correct answer.*

(1) For any set A, $A \in \mathcal{P}(A)$ [✓ T] F

(2) For all sets A and B, $\mathcal{P}(B) \subseteq \mathcal{P}(A)$ implies $B \subseteq A$ [✓ T] F

(3) For all sets A and B, $A \cup B \in \mathcal{P}(B)$ implies $A \subseteq B$ [✓ T] F

(4) $\{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\emptyset)$ T [✓ F]

(5) The relation $\{(n, n+k) | n \in \mathbb{N}, k \in \mathbb{N}^+\} =: \mathcal{R}$ is transitive [✓ T] F

(6) There exists a relation on $\mathbb{Z}$ that is both symmetric and anti-symmetric. [✓ T] F

(1) T since $A \subseteq A$

(2) T since $\mathcal{P}(B) \subseteq \mathcal{P}(A)$ means that for all sets S, $S \subseteq B$ implies $S \subseteq A$. Hence also $B \subseteq A$.

(3) $A \cup B \in \mathcal{P}(B)$ implies $A \cup B \subseteq B$, so also $A \subseteq B$.

(4) $\{\emptyset, \{\emptyset\}\} \nsubseteq \mathcal{P}(\emptyset)$ because $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $\{\emptyset\} \notin \{\emptyset\}$

(5) Take any $a, b, c \in \mathbb{N}$.
$(a,b), (b,c) \in \mathcal{R} \Rightarrow b = a + k$ and $c = b + \ell$ for $k, \ell \in \mathbb{N}^+$
$\Rightarrow c = (a+k) + \ell = a + (k+\ell)$ and $k + \ell \in \mathbb{N}^+$
$\Rightarrow (a, c) \in \mathcal{R}$.

(6) Example: $\mathcal{R} = \{(n,n) : n \in \mathbb{Z}\}$

8. Consider the following function:

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^{+}; \quad f(x, y) = (xy + x, 2^{y})$$

Which of the following statements about f are true? *Circle each correct answer below.*

(i) f is one-to-one (injective). T [✓ F]

(ii) f is onto (surjective). T [✓ F]

(iii) f is bijective. T [✓ F]

(i) e.g. $f(0,-1) = f(1,-1)$

(ii) $f(x,-1) = (0, 2^{-1})$ for all $x \in \mathbb{R}$ so there is no $(x,y) \in \mathbb{R}^2$ s.t. $f(x,y) = (1, 2^{-1})$.

9. For each of the statements below, determine whether it is true or false. *Circle each correct answer below.*

(a) Every proposition is logically equivalent to a proposition containing only the connectives ¬ and →. [✓ T] F

(b) If the complete truth tree of a proposition P has no inactive paths, then P is a tautology. T [✓ F]

(c) If the set of premises of an argument is inconsistent, then the argument is valid. [✓ T] F

(d) For any two functions $f, g : \mathbb{N} \to \mathbb{N}$, if f and g are bijective, then so is $g \circ f$. [✓ T] F

(e) For any two finite sets A, B, if $|A| < |B|$, then every function $f : B \to A$ is onto but not one-to-one. T [✓ F]

(a) $p \vee q \equiv \neg p \to q$, $p \wedge q \equiv \neg(\neg p \vee \neg q) \equiv \neg(p \to \neg q)$
the rest follows...

Short-answer questions — write your final answer in the answer box. Wherever indicated, you must briefly justify your answers to receive full marks.

10. In the following question, you do not have to justify your answers.

(a) Give a precise definition of a *transitive* relation.

Answer: A relation R on a set A is transitive if for all $a,b,c \in A$: $aRb \wedge bRc \rightarrow aRc$.

(b) Give a precise definition of a *symmetric* relation.

Answer: A relation R on a set A is symmetric if for all $a,b \in A$: $aRb \rightarrow bRa$.

(c) Give an example of a relation on the set $A = \{0, 1, 2, 3\}$ which is symmetric but not transitive.

Answer: $R = \{(0,1), (1,0), (1,2), (2,1)\}$

(d) Give an example of a relation on the set $B = \{0, 1\}$ which is symmetric but not transitive.

Answer: it does not exist

11. From a group of 12 men and 15 women, a committee consisting of 6 people is chosen. In how many ways is this possible if the committee must contain at least one, but no more than four women? *Your answer may include unevaluated factorials, binomial coefficients, powers, products, or sums.*

Answer: $\binom{15}{1}\binom{12}{5}+\binom{15}{2}\binom{12}{4}+\binom{15}{3}\binom{12}{3}+\binom{15}{4}\binom{12}{2}$

Justification: (# such committees) =

$= \sum_{i=1}^{4}$ (# committees with exactly i women)

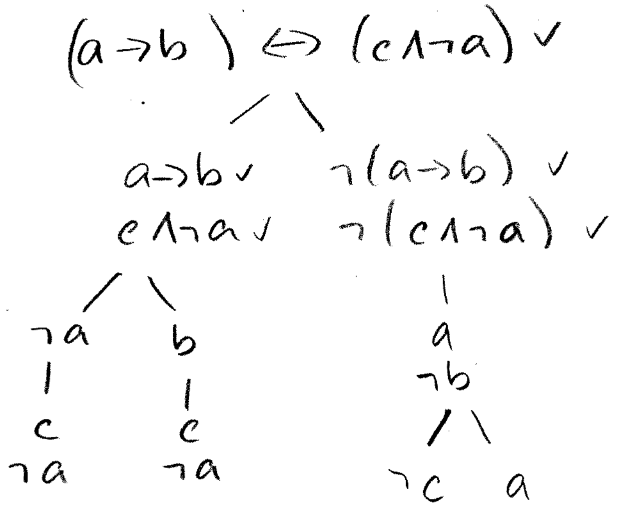
$= \binom{15}{1}\binom{12}{5}+\binom{15}{2}\binom{12}{4}+\binom{15}{3}\binom{12}{3}+\binom{15}{4}\binom{12}{2}$

12. Find a proposition in DNF equivalent to $(a \to b) \leftrightarrow (c \wedge \neg a)$.

Answer: $(\neg a \wedge c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge \neg b)$

$\equiv (\neg a \wedge c) \vee (a \wedge \neg b)$

Justification:



13. How many integers between 200 and 1000 (inclusive) are divisible by 8 or by 12?

Answer: 135

Justification:

$A = \{ n \in \mathbb{Z} : 200 \le n \le 1000, 8 | n \}$

$B = \{ n \in \mathbb{Z} : 200 \le n \le 1000, 12 | n \}$

$A \cap B = \{ n \in \mathbb{Z} : 200 \le n \le 1000, 24 | n \}$

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = \lfloor \frac{1000}{8} \rfloor - \lfloor \frac{199}{8} \rfloor = 125 - 24 = 101$

$|B| = \lfloor \frac{1000}{12} \rfloor - \lfloor \frac{199}{12} \rfloor = 83 - 16 = 67$

$|A \cap B| = \lfloor \frac{1000}{24} \rfloor - \lfloor \frac{199}{24} \rfloor = 41 - 8 = 33$

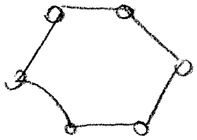
$|A \cup B| = 101 + 67 - 33 = 135$

14. Give an example (draw a figure) of simple graphs G and H such that all of the following conditions are met:

- G and H are not isomorphic
- G and H both have 6 vertices
- G and H both have 6 edges
- G and H have the same degree sequence

Answer:

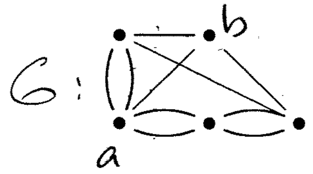
G:



H:

(No justification is needed.)

15. Does the following graph admit an Euler trail?



Answer: [✓ YES] NO

Justification: G has exactly 2 vertices of odd degree, a and b, hence it admits an Euler trail, necessarily with endpoints a and b.

16. Determine the coefficient of x^{11} in the expansion of $(2x^2 - \frac{3}{x})^{28}$. *Your answer may include unevaluated factorials, binomial coefficients, powers, products, or sums.*

Answer: $-\binom{28}{13} 2^{13} \cdot 3^{15}$

Justification: $(2x^2 - 3x^{-1})^{28} = \sum_{i=0}^{28} \binom{28}{i} (2x^2)^{28-i} (-3x^{-1})^i$

$= \sum_{i=0}^{28} \binom{28}{i} 2^{28-i} x^{56-2i} (-3)^i x^{-i} = \sum_{i=0}^{28} \binom{28}{i} 2^{28-i} (-3)^i x^{56-3i}$

Want: $56-3i = 11$ so $i=15$

coeff at x^{11}: $\binom{28}{15} 2^{28-15} (-3)^{15} = -\binom{28}{15} 2^{13} \cdot 3^{15}$

$= -\binom{28}{13} 2^{13} \cdot 3^{15}$

17. Consider the following argument:

    French fries are healthy, unless you put mayonnaise on them.
    French fries are tasty only if you put mayonnaise on them.
    Therefore, for French fries to be tasty it is necessary that they be unhealthy.

    (a) Translate this argument into propositional logic. Clearly define the propositional variables you use.

    (b) Use a truth tree to determine the validity of the argument.

(a) Define: H: "French fries are healthy"

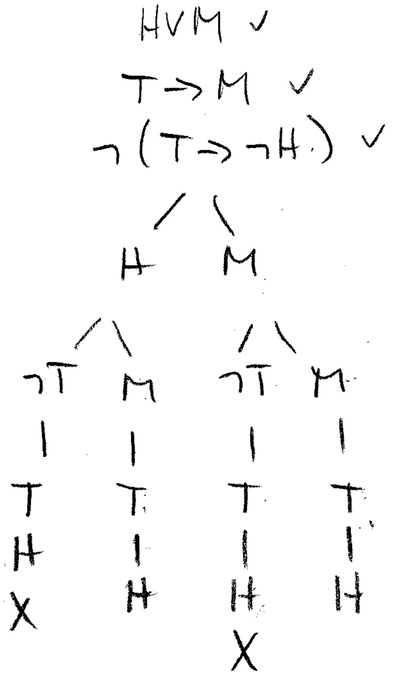
T: "French fries are tasty."

M: "You put mayo on French fries"

Argument:

$$\begin{array}{l} H \vee M \\ T \to M \\ \hline \therefore\ T \to \neg H \end{array}$$

(b)



The argument is invalid.

Counterexample:

M = true

T = true

H = true

18. Use **Mathematical Induction** to prove that for all integers $n \geq 1$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \frac{n}{2} + 1.$$

Define $P(n)$: " $1 + \frac{1}{2} + .. + \frac{1}{n} \leq \frac{n}{2} + 1$ "

Prove $P(n)$ is T for all $n \geq 1$.

BI : to prove $P(1)$: " $1 \leq \frac{1}{2} + 1$ "

LHS : 1

RHS : $\frac{1}{2} + 1 = \frac{3}{2}$ so LHS $\leq$ RHS and $P(1)$ is T.

IS : to prove $P(k) \to P(k+1)$ for all $k \geq 1$.

Fix $k \geq 1$, and assume $P(k)$: " $1 + \frac{1}{2} + .. + \frac{1}{k} \leq \frac{k}{2} + 1$ " (IH)

Examin $P(k+1)$: " $1 + \frac{1}{2} + .. + \frac{1}{k+1} \leq \frac{k+1}{2} + 1$ "

LHS: $1 + \frac{1}{2} + .. + \frac{1}{k} + \frac{1}{k+1} \overset{\text{by IH}}{\leq} \left(\frac{k}{2} + 1\right) + \frac{1}{k+1}$

$$= \left(\frac{k}{2} + \frac{1}{k+1}\right) + 1 = \frac{k^2 + k + 2}{2(k+1)} + 1 \underset{\substack{\uparrow \\ \text{since } 1 \leq k}}{\leq} \frac{k^2 + 2k + 1}{2(k+1)} + 1$$

$$= \frac{(k+1)^2}{2(k+1)} + 1 = \frac{k+1}{2} + 1$$

RHS : $\frac{k+1}{2} + 1$

Hence LHS $\leq$ RHS, and $P(k+1)$ follows from $P(k)$.

Conclusion: Since $P(1)$ is T, and $P(k) \to P(k+1)$ is T for all $k \geq 1$, by PMI, $P(n)$ is T for all $n \geq 1$.

19. Define a binary relation $\mathcal{R}$ on the set $\mathbb{Z}$ as follows:

$$x\mathcal{R}y \qquad \text{if and only if} \qquad x+y=2k \text{ for some integer } k.$$

   (a) Prove that $\mathcal{R}$ is an equivalence relation.

   (b) Describe the equivalence classes of $\mathcal{R}$. How many distinct equivalence classes are there?

(a) R is reflexive: take any $x \in \mathbb{Z}$.
Then $x+x=2x$ and $x \in \mathbb{Z}$, so xRx.

R is symmetric: take any $x, y \in \mathbb{Z}$.

$xRy \Rightarrow x+y=2k$ for some $k \in \mathbb{Z}$

$\Rightarrow y+x=2k$ and $k \in \mathbb{Z}$ $\Rightarrow yRx$

R is transitive: take any $x, y, z \in \mathbb{Z}$

xRy and yRz $\Rightarrow$ $x+y=2k$ and $y+z=2l$ for some $k, l \in \mathbb{Z}$

$\Rightarrow x+z=(x+y)+(y+z)-2y=2k+2l-2y=2(k+l-y)$.
and $k+l-y \in \mathbb{Z}$

$\Rightarrow xRz$

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

(b) $[x]=\{y \in \mathbb{Z} : x+y=2k \text{ for some } k \in \mathbb{Z}\}$

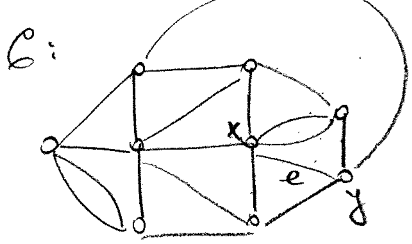
$=\{y \in \mathbb{Z} : x \equiv y \pmod 2\}$

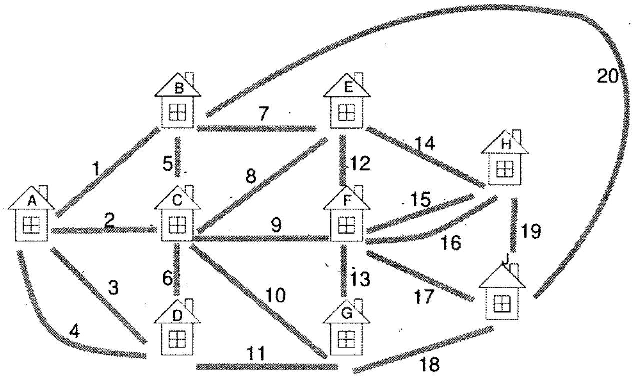
there are 2 equivalence classes:

$[0]=\{y \in \mathbb{Z} : y \text{ is even}\}$

$[1]=\{y \in \mathbb{Z} : y \text{ is odd}\}$

20. Consider the following village (the grey lines indicate roads between houses).





(a) Is it possible to take a walk through the village in such a way that you use every road exactly once? Cite appropriate theorems from graph theory to support your answer.

(b) Is it possible for such a walk to start and end at the same house?

(c) Suppose road number 17 is closed. Can we take a walk using all the remaining roads exactly once?

(a) Yes: the graph G that models this problem has no vertices of odd degree, hence G has an Euler tour

(b) Yes: G has an E. tour but not an E. trail (the latter requires exactly 2 vertices of odd degree). Hence such a walk will necessarily start and end at the same vertex (house)

(c) Yes: graph G−e has exactly 2 vertices of odd degree, so it admits an Euler trail, necessarily with endpoints x and y.

21. Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be arbitrary functions.

(i) Suppose that $g \circ f$ is one-to-one (injective). Is g then necessarily one-to-one as well? Prove or give a counterexample.

No. E.g. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$ and

$g(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Then $(g \circ f)(x) = \begin{cases} x & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$ and $g \circ f$ is one-to-one. However, g is not as $g(1) = g(0) = 1$

(ii) For any subset $S \subseteq \mathbb{Z}$, denote as usual $f(S) = \{f(x) : x \in S\}$. Prove that for any $A, B \subseteq \mathbb{Z}$, we have

$$f(A \cup B) = f(A) \cup f(B).$$

To prove $f(A \cup B) \subseteq f(A) \cup f(B)$:

Take any $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. If $x \in A$, then $y = f(x) \in f(A)$, and if $x \in B$, then $y = f(x) \in f(B)$. Hence $y \in f(A) \cup f(B)$.

To prove $f(A) \cup f(B) \subseteq f(A \cup B)$:

Take any $y \in f(A) \cup f(B)$. Then $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then $y = f(a)$ for some $a \in A$, and if $y \in f(B)$, then $y = f(b)$ for some $b \in B$. In either case, $y = f(x)$ for some $x \in A \cup B$, so $y \in f(A \cup B)$.

We conclude that $f(A \cup B) = f(A) \cup f(B)$.