MECH 4101 - Mechanics of Deformable Solids Notes

1.0 Elastic Theory

1.1 Stress

Stress is an internal resistance of a material and is defined as an average force exerted over an area. Two types of stresses: normal stress and shear stress arise due to the force acting normal and parallel to the plane, respectively. Mathematically, stress can be written as:

$$\sigma = \frac{F_\mathrm{N}}{A} \tag{1.1}$$

$$\tau = \frac{F_\mathrm{T}}{A} \tag{1.2}$$

where:

- σ is normal stress
- τ is shear stress
- F_N is normal force
- F_T is tangential force
- A is cross-sectional area

A double-suffix notation, τ_xy, is used to define the direction of shear stress acting on a plane. The first suffix defines the direction of the outward normal of the plane and the second one represents the direction of the shear stress. It can be shown that:

$$\tau_\mathrm{xy} = \tau_\mathrm{yx}; \quad \tau_\mathrm{yz} = \tau_\mathrm{zy}; \quad \tau_\mathrm{xz} = \tau_\mathrm{zx} \tag{1.3}$$

In the 3-D case, the stress matrix $[\sigma]$ can be written as:

$$[\sigma] = \begin{bmatrix} \sigma_\mathrm{xx} & \tau_\mathrm{xy} & \tau_\mathrm{xz} \\ \tau_\mathrm{xy} & \sigma_\mathrm{yy} & \tau_\mathrm{yz} \\ \tau_\mathrm{xz} & \tau_\mathrm{yz} & \sigma_\mathrm{zz} \end{bmatrix} \tag{1.4}$$

In solid mechanics, strain is a measure of deformation due to the applied load with reference to the original length. The 3-D normal strain in a solid medium due to an applied load is given by:

$$\varepsilon_\mathrm{xx} = \frac{\partial u}{\partial x} \tag{1.5}$$

$$\varepsilon_\mathrm{yy} = \frac{\partial v}{\partial y} \tag{1.6}$$

$$\varepsilon_\mathrm{zz} = \frac{\partial w}{\partial z} \tag{1.7}$$

where ε_xx, ε_yy, and ε_zz are normal strains in the x, y and z directions, respectively and u , v , and w are displacements in the x, y, and z directions, respectively.

In addition to normal strain, shear strain, γ, due to shear force is given by:

$$\gamma_{\mathrm{xy}} = \gamma_{\mathrm{yx}} \Rightarrow \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\varepsilon_{\mathrm{xy}} \tag{1.8}$$

$$\gamma_{\mathrm{zy}} = \gamma_{\mathrm{yz}} \Rightarrow \frac{\partial w}{\partial y} + \frac{\partial v}{\partial w} = 2\varepsilon_{\mathrm{yz}} \tag{1.9}$$

$$\gamma_{\mathrm{xz}} = \gamma_{\mathrm{zx}} \Rightarrow \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 2\varepsilon_{\mathrm{xz}} \tag{1.10}$$

Six independent stress components σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{yz}, and τ_{zx} are needed to describe the load carried by a 3-D solid at any point. The six independent strain components $\varepsilon_{\mathrm{xy}}$, $\varepsilon_{\mathrm{yy}}$, $\varepsilon_{\mathrm{zz}}$, γ_{xy} ,γ_{yz}, γ_{zx}, which are produced by the load, are used to describe the deformation at any point.

1.2 Equation of Equilibrium

Consider an infinitesimal element $\Delta x \times \Delta y \times \Delta z$ onto which non-uniform stresses are applied at different faces, as shown in Figure 1.

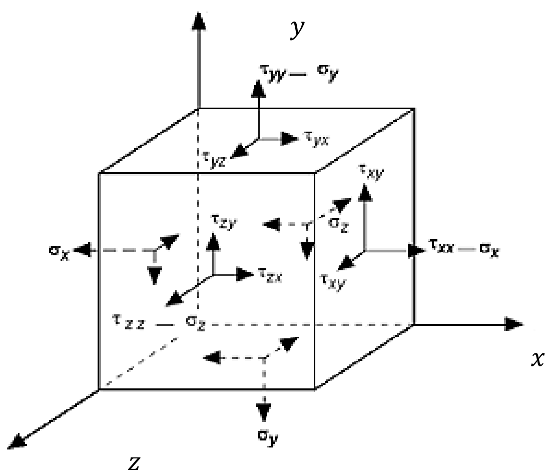


Figure 1: An infinitesimal element showing stress distribution.

$\sum F_{\mathrm{x}} = 0$, gives

$$(\sigma_{\mathrm{xx}} + \Delta\sigma_{\mathrm{xx}})\Delta y \Delta z - \sigma_{\mathrm{xx}}\Delta y \Delta z + \left(\tau_{\mathrm{yx}} + \Delta\tau_{\mathrm{yx}}\right)\Delta x \Delta z - \Delta\tau_{\mathrm{yx}}\Delta x \Delta z + (\tau_{\mathrm{zx}} + \Delta\tau_{\mathrm{zx}})\Delta x \Delta y - \Delta\tau_{\mathrm{zx}}\Delta x \Delta y = 0 \tag{1.11}$$

Dividing both sides by $\Delta x \Delta y \Delta z$ and taking the limit as $\Delta x, \Delta y,$ and Δz goes to zero, (1.11) can be rewritten as:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{1.12}$$

In the presence of body forces, F_x, (1.12) can be rewritten as:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \tag{1.13}$$

Similarly for $\sum F_y = 0$:

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \tag{1.14}$$

And for $\sum F_z = 0$:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0 \tag{1.15}$$

1.3 Stress-Strain Relationship

In general for a 3-D isotropic solid, the stress-strain relationship can be written as:

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \tag{1.16}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \tag{1.17}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \tag{1.18}$$

E is Young's modulus and ν is Poisson's ratio

Shear stress is related to the shear strain by:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}; \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}; \qquad \gamma_{xz} = \frac{\tau_{xz}}{G}; \tag{1.19}$$

where shear stress is given by:

$$G = \frac{E}{2(1+\nu)} \tag{1.20}$$

1.4 Vectors and Tensors Notation

In order to represent stresses, strains using tensor notation, it is important to set up the index notation using Einstein's summation convention. This notation is used to simplify expressions involving vectors and tensors operations.

The direction *x*, *y*, and *z* is referred to as 1, 2, and 3, respectively.

A vector $\vec{v}$ in the index notation can be written as;

$$\vec{v} = v_i \Rightarrow \sum_{i=1}^{3} v_i \vec{e_i} \quad (1.21)$$

where $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ are the orthonormal unit vectors in 3-D space

Similarly, a tensor in the index notation can be written as:

$$T \Rightarrow T_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\vec{e_i}\vec{e_j} \quad (1.22)$$

where $\{\vec{e_i}, \vec{e_j}\}$ are unit dyads that represent an ordered pair of coordinate directions

In Einstein's summation convention, if the index appears more than once, then the summation is repeated for each index over the range of the index. The number of free indices determine whether the index notation is a scalar, a vector, or a tensor. In this method, no free index or rank 0, represents a scalar, one free index or rank 1, means it is a vector, two free indices or rank 2 represents a tensor, and so on. For the Cartesian co-ordinate system where the summation is performed in three mutually- orthogonal space, the rank of the tensor n have 3^n elements.

For example:

$\Rightarrow u_i v_i, T_{ii}, \dfrac{\partial u_i}{\partial x_i}$, and $\dfrac{\partial^2 f}{\partial x_i \partial x_i}$ are scalars (no free index, rank 0 = 1 element)

$\Rightarrow \epsilon_{ijk}, u_i v_j, \dfrac{\partial f}{\partial x_i}$, and $T_{ii} v_i$ are vectors (one free index, rank 1 = 3 elements)

$$\Rightarrow \epsilon_{ijk} \begin{cases} +1 \text{ when any two indices i, j, k are equal} \\ -1 \text{ when ijk} = 1\ 2\ 3 \text{ or } 3\ 2\ 1 \text{ or } 2\ 3\ 1 \\ 0 \text{ otherwise} \end{cases}$$

$\Rightarrow u_i v_j$, and $\dfrac{\partial u_j}{\partial x_i}$, and $\dfrac{\partial^2 f}{\partial x_i \partial x_i}$ are tensors (two free indices, rank 2 = 9 element)

$\Rightarrow c_{ijk}$ is a tensor (four free indices, rank 4 = 81 elements)

Based on the aforementioned conventions, the strain tensor can be written as:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (1.23)$$

The stress tensor is given by:

$$T_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{1.24}$$

The relationship of strain as a function of stress can be written as:

$$\varepsilon_{ij} = \frac{1+\nu}{E} T_{ij} - \frac{\nu}{E}(T_{11} + T_{22} + T_{33})\delta_{ij} \tag{1.25}$$

where, $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ also known as Kronecker delta

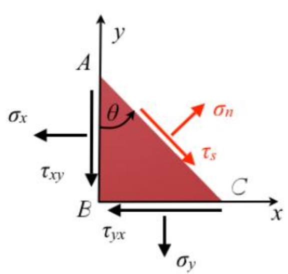
and the stress in terms of strain is given by:

$$T_{ij} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\delta_{ij} + 2G\varepsilon_{ij} \tag{1.26}$$

Finally, the equilibrium equations for elasticity can be written as:

$$\frac{\partial T_{ij}}{\partial x_i} + F = 0 \tag{1.27}$$

1.5 Stress at a Point



Transforming the coordinate system by angle θ and knowing $\tau_{yx} = \tau_{xy}$:

$$\sigma_n(AC) + [\tau_{yx}(BC)\cos\theta] - \sigma_x(AB)\cos\theta + \tau_{yx}(AB)\sin\theta - \sigma_y(BC)\sin\theta = 0$$

Divide every term by area AC:

$$\sigma_n + \left[\tau_{yx}\left(\frac{BC}{AC}\right)\cos\theta\right] - \sigma_x\left(\frac{AB}{AC}\right)\cos\theta + \tau_{yx}\left(\frac{AB}{AC}\right)\sin\theta - \sigma_y\left(\frac{BC}{AC}\right)\sin\theta = 0$$

where:

$$\frac{BC}{AC} = \sin\theta \text{ and } \frac{AB}{AC} = \cos\theta \text{; areas are constant}$$

It can be further simplified to:

$$\sigma_n + \tau_{yx} \sin\theta \cos\theta - \sigma_x \cos^2\theta \ + \tau_{yx} \cos\theta \sin\theta - \sigma_y \sin^2\theta = 0$$
$$\Rightarrow \sigma_n - \sigma_x \cos^2\theta - \sigma_y \sin^2\theta + 2\tau_{yx} \cos\theta \sin\theta$$

For Mohr's circle, if the shear stress tends a clockwise (C.W.) rotation, the element is positive.

Using:

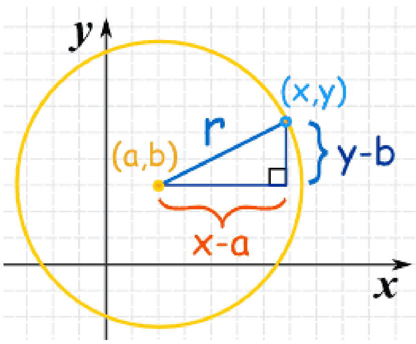
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2};\ \cos^2\theta = \frac{1 + \cos 2\theta}{2}; 2\sin\theta + \cos\theta = \sin 2\theta$$

$$\sigma_n = \frac{1}{2}(\sigma_x - \sigma_y) + \frac{1}{2}(\sigma_x + \sigma_y)\cos 2\theta + \tau_{yx} \sin\theta \qquad ①$$
$$\tau_s = \frac{1}{2}(\sigma_x + \sigma_y)\sin 2\theta - \tau_{xy} \cos 2\theta \qquad ②$$

Squaring ① and ② and adding them:

$$\tau_s = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2$$

Comparing this to the equation of a circle:

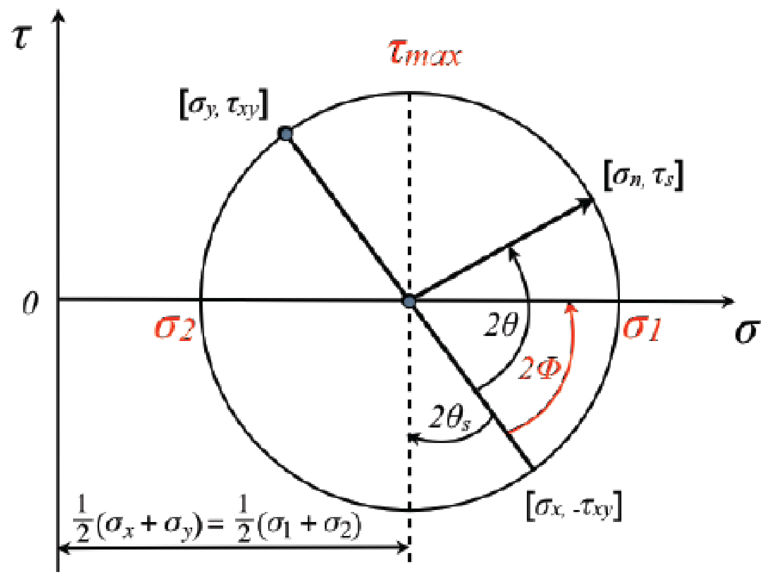


$$(x - a)^2 + (y - b)^2 = r^2$$

...so:

$$r = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2$$

Hence the **Mohr's circle**:



$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_n = \frac{1}{2}(\sigma_x - \sigma_y) + \frac{1}{2}(\sigma_x + \sigma_y)\cos 2\theta + \tau_{yx}\sin\theta$$

$$\tau_s = \frac{1}{2}(\sigma_x + \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\Phi = \frac{\tau_{xy}}{\frac{1}{2}(\sigma_x - \sigma_y)}$$

$$|\tau_{max}| = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$|\tau_{max}| = \frac{\sigma_x - \sigma_y}{2}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Principals of Solutions in Solid Mechanics:

i) condition of state equilibrium
ii) stress-strain relationship
iii) condition for compatibility
iv) boundary conditions (BCs)

Methods of Solution

a) engineering (strength of material) approach
   - simplifying assumptions related to the geometry of deformation, the state of distribution of stress/strain across the cross section
   - useful in the simple beam bending and torsion problem
b) "exact" – mathematical solution to the governing equations
c) numerical solution (FEM, BEM)
d) experiments

Compatibility Equation in Terms of Stress

$$\left(\frac{\partial}{\partial \mathrm{x}^2}+\frac{\partial}{\partial \mathrm{y}^2}\right)(\sigma_\mathrm{x}+\sigma_\mathrm{y})=0$$

$$\nabla^2(\sigma_\mathrm{x}+\sigma_\mathrm{y})=0$$

True when: $F_\mathrm{x}, F_\mathrm{y}$, and T are zero.

Plane strain ($\varepsilon_\mathrm{z}=0$):

$$\nabla^2(\sigma_\mathrm{x}+\sigma_\mathrm{y}+\mathrm{E}\alpha T)=-\frac{1}{1-\nu}\left(\frac{\partial F_\mathrm{x}}{\partial \mathrm{x}}+\frac{\partial F_\mathrm{y}}{\partial \mathrm{y}}\right)$$

Plane stress ($\sigma_\mathrm{z}=0$):

$$\nabla^2(\sigma_\mathrm{x}+\sigma_\mathrm{y}+\mathrm{E}\alpha T)=-(1+\nu)\left(\frac{\partial F_\mathrm{x}}{\partial \mathrm{x}}+\frac{\partial F_\mathrm{y}}{\partial \mathrm{y}}\right)$$

2.0 Elastic Axis-symmetric Deformations

- axis-symmetric problems in stress analysis occur in numerous practical-pressure vessels, compound cylinders, turbine/compressor disks, flywheels
- not only geometry has to be similar, loads applied along an axis has to be similar

2.1 Basic Relation

- general transformation of Cartesian to cylindrical polar

$$x=r\cos\theta$$

$$y=r\sin\theta$$

$$x^2+y^2=r^2$$

$$\tan\theta=\frac{y}{x}$$

$$2r\frac{\mathrm{d}r}{\mathrm{d}x}=2x\Rightarrow\frac{\mathrm{d}r}{\mathrm{d}x}=\cos\theta$$

$$2r\frac{\mathrm{d}r}{\mathrm{d}y}=2y\Rightarrow\frac{\mathrm{d}r}{\mathrm{d}y}=\sin\theta$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=-\frac{y}{r^2}\Rightarrow-\frac{\sin\theta}{r}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}y}=-\frac{x}{r^2}\Rightarrow\frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial \mathrm{x}}=\frac{\partial r}{\partial \mathrm{x}}\frac{\partial}{\partial \mathrm{r}}+\frac{\partial\theta}{\partial \mathrm{x}}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial \mathrm{y}}=\frac{\partial r}{\partial \mathrm{y}}\frac{\partial}{\partial \mathrm{r}}+\frac{\partial\theta}{\partial \mathrm{y}}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial \mathrm{x}}=\cos\theta\frac{\partial}{\partial \mathrm{r}}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial \mathrm{y}}=\sin\theta\frac{\partial}{\partial \mathrm{r}}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

In matrix form:

$$\begin{bmatrix} \frac{\partial}{\partial \mathrm{x}} \\ \frac{\partial}{\partial \mathrm{y}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial \mathrm{r}} \\ \frac{\partial}{\partial \theta} \end{bmatrix}$$

- for axis-symmetric, all displacements are $f(r)$ and independent of $\theta \Rightarrow r = 0$ (pressurized cylinders and rotating disks)
- shear in $\tau_{\mathrm{r}\theta} = 0$ (principle stress) $\Rightarrow \sigma_\mathrm{r}, \sigma_\theta$ are $f(r)$

For equilibrium:

$$\frac{\mathrm{d}\sigma_\mathrm{r}}{\mathrm{d}r} + \frac{\sigma_\mathrm{r} - \sigma_\theta}{r} + F_\mathrm{r} = 0$$

Strain-Displacement Relationship

$$\left.\begin{aligned} \varepsilon_\mathrm{r} &= \frac{\mathrm{d}u}{\mathrm{d}r} \\ \varepsilon_\theta &= \frac{u}{r} \end{aligned}\right\} 2.2$$

Hooke's Law

$$\varepsilon_\mathrm{r} = \frac{1}{\mathrm{E}}[\sigma_\mathrm{r} - \nu(\sigma_\theta + \sigma_\mathrm{z})] + \alpha\Delta T$$

$$\varepsilon_\theta = \frac{1}{\mathrm{E}}[\sigma_\theta - \nu(\sigma_\mathrm{r} + \sigma_\mathrm{z})] + \alpha\Delta T$$

$$\varepsilon_\mathrm{z} = \frac{1}{\mathrm{E}}[\sigma_\mathrm{z} - \nu(\sigma_\mathrm{r} + \sigma_\theta)] + \alpha\Delta T$$

Plane stress ($\boldsymbol{\sigma_\mathrm{z} = 0}$)

(thin rotating disk; open-ended cylinder)

$$\sigma_\mathrm{r} = \frac{\mathrm{E}}{1 - \nu^2}[\varepsilon_\mathrm{r} + \nu\varepsilon_\theta(1 + \nu)\alpha T]$$

$$\sigma_\theta = \frac{\mathrm{E}}{1 - \nu^2}[\varepsilon_\theta + \nu\varepsilon_\mathrm{r}(1 + \nu)\alpha T]$$

Plane strain ($\boldsymbol{\varepsilon_\mathrm{z} = 0}$)

(long cylinders with restrained ends)

$$\sigma_\mathrm{z} = \nu(\sigma_\mathrm{r} - \sigma_\theta) - \mathrm{E}\alpha T$$

$$\varepsilon_\mathrm{r} = \frac{1 - \nu^2}{\mathrm{E}}\left[\sigma_\mathrm{r} - \frac{\nu}{1 - \nu}\sigma_\theta\right] + (1 + \nu)\alpha T$$

$$\varepsilon_\theta = \frac{1 - \nu^2}{\mathrm{E}}\left[\sigma_\theta - \frac{\nu}{1 - \nu}\sigma_\mathrm{r}\right] + (1 + \nu)\alpha$$

2.2 Pressurized and Thick-Walled Cylinders

Under pressure loads, the stresses in the cylinder are given by Lamé's solution:

$$\sigma_\mathrm{r} = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

where A and B are found from the given boundary conditions.

Example 1: Thick-walled open-ended cylinder subject to internal pressure: $p = 100$ MPa. Find the maximum stress in the cylinder given:

$$r_\mathrm{i} = 25\ \mathrm{mm}$$

$$r_\mathrm{o} = 50\ \mathrm{mm}$$

Boundary conditions:

@ $r = r_\mathrm{i}$

$$\sigma_\mathrm{r} = A - \frac{B}{r_\mathrm{i}^2}$$

$$-p = A - \frac{B}{(25\ \mathrm{mm})^2} \quad ①$$

($-p$ due to compression)

@ $r = r_\mathrm{o}$

$$\sigma_\mathrm{r} = A - \frac{B}{r_\mathrm{o}^2}$$

$$0 = A - \frac{B}{(50\ \mathrm{mm})^2} \quad ②$$

② → ①

$$-100\ \mathrm{MPa} = \frac{B}{(50 \times 10^{-3}\ \mathrm{m})^2} - \frac{B}{(25 \times 10^{-3}\ \mathrm{m})^2} \Rightarrow -100\ \mathrm{MPa} = \frac{B}{0.0025\ \mathrm{m}^2} - \frac{B}{6.25 \times 10^{-4}\ \mathrm{m}^2}$$

$$\Rightarrow -100\ \mathrm{MPa} = \frac{(6.25 \times 10^{-4}\ \mathrm{m}^2)B - (0.0025\ \mathrm{m}^2)B}{1.56 \times 10^{-6}\ \mathrm{m}^4}$$

$$\Rightarrow -156.25\ \mathrm{N \cdot m^2} = (-1.88 \times 10^{-4}\ \mathrm{m}^2)B$$

$$\Rightarrow \boldsymbol{B = 8.33 \times 10^4\ \mathrm{N}}$$

$$0 = A - \frac{B}{(50\ \mathrm{mm})^2} \Rightarrow A = \frac{8.33 \times 10^{-4}\ \mathrm{N}}{25 \times 10^{-3}\ \mathrm{m}^2}$$

$$\Rightarrow \boldsymbol{A = 3.33 \times 10^7\ \mathrm{Pa}}$$

$$\sigma_\mathrm{r} = 3.33 \times 10^7\ \mathrm{Pa} - \frac{8.33 \times 10^4\ \mathrm{N}}{r^2}$$

$$\sigma_\theta = 3.33 \times 10^7\ \mathrm{Pa} + \frac{8.33 \times 10^4\ \mathrm{N}}{r^2}$$

$$\boldsymbol{(\sigma_\mathrm{r})_{max} = -100\ \mathrm{MPa}}$$

$$\boldsymbol{\sigma_\theta @ r = r_\mathrm{i} \Rightarrow 166.6\ \mathrm{MPa}}$$

$$\boldsymbol{\sigma_\theta @ r = r_\mathrm{o} \Rightarrow 66.6\ \mathrm{MPa}}$$

In general, the **boundary conditions** are:

$$\left.\begin{aligned} r = r_\mathrm{i} \Rightarrow \sigma_{r_\mathrm{i}} = -p_\mathrm{i} \Rightarrow -p_\mathrm{i} = A - \frac{B}{r_\mathrm{i}^2} \\ r = r_\mathrm{o} \Rightarrow \sigma_{r_\mathrm{o}} = -p_\mathrm{o} \Rightarrow -p_\mathrm{o} = A - \frac{B}{r_\mathrm{o}^2} \end{aligned}\right\} \begin{aligned} A = \frac{p_\mathrm{i} - p_\mathrm{o}k^2}{k^2 - 1} \\ B = \frac{(p_\mathrm{i} - p_\mathrm{o})r_\mathrm{o}^2}{k^2 - 1} \end{aligned} \quad ;\text{where } k = \frac{r_\mathrm{o}}{r_\mathrm{i}}$$

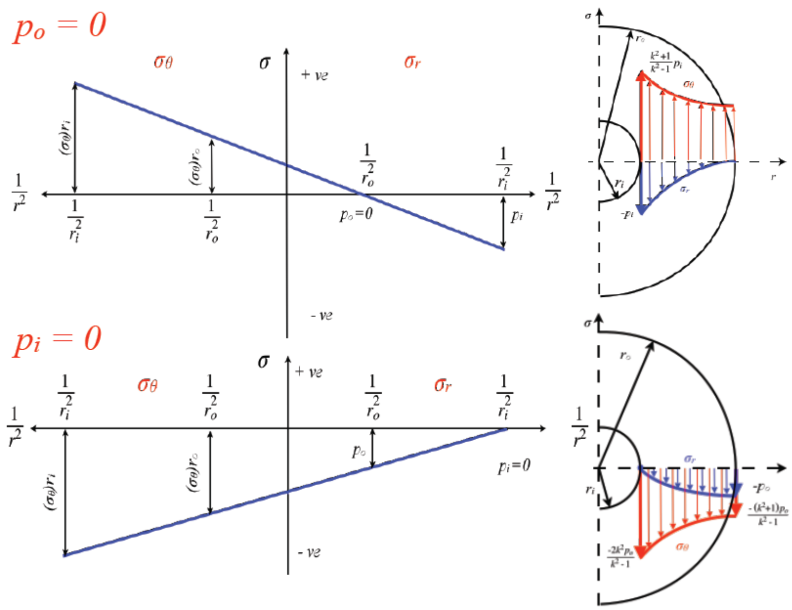
For **internal pressure** ONLY ($p_\mathrm{o} = 0$):

$$\sigma_\mathrm{r} = \frac{p_\mathrm{i}}{k^2 - 1}\left(1 - \frac{r_\mathrm{o}^2}{r^2}\right);\ \sigma_\theta = \frac{p_\mathrm{i}}{k^2 - 1}\left(1 + \frac{r_\mathrm{o}^2}{r^2}\right)$$

For **external** loading ONLY ($p_\mathrm{i} = 0$):

$$\sigma_\mathrm{r} = \frac{p_\mathrm{i}k^2}{k^2 - 1}\left(\frac{r_\mathrm{i}^2}{r^2} - 1\right);\ \sigma_\theta = \frac{-p_\mathrm{o}}{k^2 - 1}\left(1 - \frac{r_\mathrm{o}^2}{r^2}\right)$$

$$\sigma_\theta > \sigma_z > \sigma_r$$



Example: A <u>long</u>, <u>thick-walled</u> cylinder with <u>ends rigidly restrained</u> is embedded in a rigid wall and subjected to an internal pressure of p_i. What is the maximum p allowed to avoid yielding according to Tresca's criteria?

$$\left(\text{Tresca: } \frac{\sigma_\theta + \sigma_r}{2} = \frac{\sigma_y}{2}\right)_{r=r_o}$$

$r_i = 20 \times 10^{-3}$ m
$r_o = 50 \times 10^{-3}$ m
$\nu = 0.33$
$E = 60$ GPa
$\sigma_y = 80$ MPa
$\frac{1}{r_o^2} = 400; \ \frac{1}{r_i^2} = 2500$

$\varepsilon_z = 0 \Rightarrow \frac{1}{E}[\sigma_z - \nu(\sigma_r - \sigma_\theta)]$
$\sigma_z = \nu(\sigma_\theta + \sigma_r)$
$(\varepsilon_\theta)_{r_o} = 0 \Rightarrow \frac{u}{r_o} = 0$; u is the displacement
$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)]_{r_o} \Rightarrow 0$

let: $a = (\sigma_\theta)_{r_i}$ & $b = (\sigma_\theta)_{r_o}$

@ r_i:
$\frac{a + P_i}{2} = \frac{80}{2}$
$\Rightarrow a + P_i = 80$ MPa ①

@ r_o:
$\sigma_\theta = b, \sigma_r = -P_o$
$u_{r_2} = 0$

$$\frac{a + P_i}{2500\ \text{m}^{-2} + 400\ \text{m}^{-2}} = \frac{P_i - P_o}{2500\ \text{m}^{-2} + 400\ \text{m}^{-2}}$$
$$\Rightarrow \frac{b + P_o}{400\ \text{m}^{-2} + 400\ \text{m}^{-2}}$$

$a = 0.8125P_1$
$P_i = 2.3125P_2$

$P_o = 25.6$ MPa

$b - \nu[-P_o \nu(b - P_o)] = 0$

$b = -\frac{\nu}{1-\nu} P_o \Rightarrow -0.5 P_o$ ②

$P_i = 59.2$ MPa

$$\boldsymbol{a = (\sigma_\theta)_{r_1} \Rightarrow 20.8\ \mathrm{MPa}}$$
$$\boldsymbol{b = (\sigma_\theta)_{r_2} \Rightarrow -12.8\ \mathrm{MPa}}$$

2.3 Rotating Disks and Long, Rotating Cylinders (T=0)

- similar to analysis as in the Lamé's equation, for pressurized cylinders, except now due to rotation there is a body force, F_r

Plane stress case (i.e. disk):

$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8} \rho r^2 \omega^2$$
$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8} \rho r^2 \omega^2$$

A and B are constants for a given geometry and speed (found using boundary conditions)

Example: A <u>thin steel disk</u> with $r_i = 80$ mm, $r_o = 380$ mm is shrink-fitted onto a <u>rigid shaft</u>. The radial interference ($\delta = 0.1$ mm). At what speed will the <u>shrink fit loosen</u>?

$E = 200$ GPa

$\nu = 0.3$

$\rho = 7\,850 \frac{\mathrm{kg}}{\mathrm{m}^3}$

When the shrink fit loosens, $\sigma_r = 0$ @ $r = r_i$

@ $r = r_o \Rightarrow \sigma_{(r_o)_2} = 0$

$$\left.\begin{aligned} &@\ r = r_i \\ &0 = A - \frac{B}{(0.08\ \mathrm{m})^2} - \left(\frac{3+0.3}{8}\right)\left(7\,850 \frac{\mathrm{kg}}{\mathrm{m}^3}\right)(0.08\ \mathrm{m})^2 \omega^2 \ \text{①} \\ &@\ r = r_o \\ &0 = A - \frac{B}{(0.38\ \mathrm{m})^2} - \left(\frac{3+0.3}{8}\right)\left(7\,850 \frac{\mathrm{kg}}{\mathrm{m}^3}\right)(0.38\ \mathrm{m})^2 \omega^2 \ \text{②} \end{aligned}\right\} \begin{aligned} A &= 488.3\omega^2 \\ B &= 2.933\omega^2 \end{aligned}$$

$$\varepsilon_\theta = \frac{u}{r} \Rightarrow \frac{r}{\mathrm{E}}(\sigma_\theta - \nu\sigma_r)$$

$$\delta = (u)_{rm_o} - (u)_{rm_i} \Rightarrow 0.1\ \mathrm{m}$$

σ_θ @ $r = r_1$

$$\frac{(0.1 \times 10^{-3}\ \mathrm{m})(200 \times 10^{9}\ \mathrm{Pa})}{(0.08\ \mathrm{m})^2} = A + \frac{B}{(0.08\ \mathrm{m})^2} - \left(\frac{3+0.3}{8}\right)\left(2\,830\frac{\mathrm{kg}}{\mathrm{m}^3}\right)(0.08\ \mathrm{m})^2\omega^2$$

$$\boldsymbol{\omega = 514.6\frac{\mathbf{rad}}{\mathbf{s}} \Rightarrow \omega = 4\,914\ \mathbf{RPM}}$$

Plane strain (long rotating cylinders)

$$\sigma_\mathrm{r} = A - \frac{B}{r^2} - \frac{3+\nu^*}{8}\rho r^2\omega^2 \qquad \nu^* = \frac{\nu}{1-\nu}$$
$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu^*}{8}\rho r^2\omega^2$$

Example:

A long, <u>solid steel shaft</u> of 200 mm radius is rotating @ 300 RPM with its <u>ends restrained</u> from longitudinal displacement. Find the total longitudinal force exerted at the ends.

$\mathrm{E} = 200\ \mathrm{GPa}, \quad \nu = 0.3, \quad \rho = 7\,850\frac{\mathrm{kg}}{\mathrm{m}^3}$

$\nu^* = \frac{0.3}{0.7} \Rightarrow 0.42857$

<u>B. Cs.</u>

$\varepsilon_\mathrm{z} = 0$

$$\varepsilon_\mathrm{z} = \frac{1}{\mathrm{E}}[\sigma_\mathrm{z} - \nu(\sigma_\mathrm{r} - \sigma_\theta)] + \alpha\Delta T \Rightarrow \boldsymbol{\sigma_\mathrm{z} = \nu(\sigma_\mathrm{r} - \sigma_\theta)}$$

$B = 0$ for a solid steel shaft

$$\sigma_\mathrm{z} = \nu\left[\left(A - \frac{B}{r^2} - \frac{3+\nu^*}{8}\rho r^2\omega^2\right) + \left(A + \frac{B}{r^2} - \frac{1+3\nu^*}{8}\rho r^2\omega^2\right)\right]$$

$$\Rightarrow \nu\left[\left(A - \frac{3+\nu^*}{8}\rho r^2\omega^2\right) + \left(A - \frac{1+3\nu^*}{8}\rho r^2\omega^2\right)\right] \Rightarrow \boldsymbol{\sigma_\mathrm{z} = \nu\left[2A - \frac{1+\nu^*}{2}\rho r^2\omega^2\right]}$$

r @ $r_\mathrm{o} = 200\ \mathrm{mm}, \qquad \sigma_\mathrm{r} = 0$

$$\sigma_\mathrm{r} = A - \frac{B}{r^2} - \frac{3+\nu^*}{8}\rho r^2\omega^2 \Rightarrow \boldsymbol{A = \frac{3+\nu^*}{8}\rho r^2\omega^2}$$

$$\sigma_\mathrm{z} = \nu\left[\frac{3+\nu^*}{4}r_\mathrm{o}^2 - \left(\frac{1+\nu^2}{2}\right)\pi^2\right]\rho\omega^2 \Rightarrow \frac{\nu\rho\omega^2}{4(1-\nu)}\ [(3-2\nu)r_\mathrm{o}^2 - 2\pi^2]$$

$$\sigma = \frac{F}{A} \Rightarrow F = \sigma A \Rightarrow F = \int \sigma \, \mathrm{d}A \Rightarrow \int_0^{2\pi}\int_0^{\pi} \sigma_z r \, \mathrm{d}r\mathrm{d}\theta \Rightarrow \frac{\pi \nu \rho \omega^2 r_o^4}{2}$$

$$\Rightarrow \frac{\pi(0.3)\left(7\,850\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(300\,\frac{\mathrm{rev}}{\mathrm{min}} \times \frac{2\pi\mathrm{rad}}{\mathrm{rev}} \times \frac{\mathrm{min}}{60\ \mathrm{s}}\right)^2 (0.2\ \mathrm{m})^4}{2} \Rightarrow \boldsymbol{F = 5.84\ \mathrm{kN}}$$

2.4 Thermal Stresses in Thick-Walled Cylinder and Disks

In linear-elastic analysis, the effects of mechanical and thermal loads can be treated separately and then added together after the components are subjected to both types of loading superposition).

- consider thermal loading ONLY

Fourier' law of heat conduction:

$$Q = -2\pi r k \frac{\mathrm{d}T}{\mathrm{d}r}$$

where:

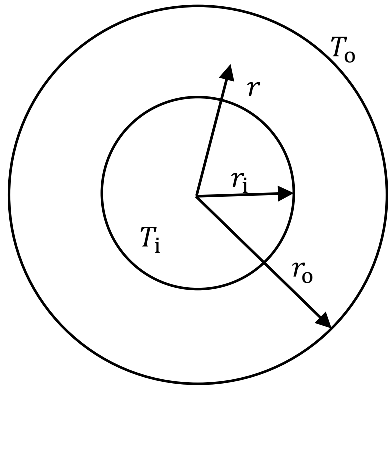
Q = heat flow per unit axial length in the radial direction

k = thermal conductivity coefficient

$$r\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{\theta}{2\pi k} \Rightarrow \mathrm{C}_1$$

$$\int \mathrm{d}T = \int \frac{\mathrm{d}r}{r} \mathrm{C}_1 \Rightarrow T = \mathrm{C}_1 \ln r + \mathrm{C}_2$$

where C_1 and C_2 are constants



B. Cs.

$@\, r = r_i \Rightarrow T = T_i$
$@\, r = r_o \Rightarrow T = T_o$

$$T = T_i + \frac{T_o - T_i}{\ln \frac{r_o}{r_i}} \ln \frac{r}{r_i}$$

2.4.1 Plane stress

- thin disks (externally cooled and heated)

$$\sigma_r = \frac{E}{1-\nu^2}[\varepsilon_r + \nu\varepsilon_\theta(1+\nu)\alpha T] \xRightarrow{\varepsilon_r=\frac{du}{dr};\ \varepsilon_\theta=\frac{u}{r}} \sigma_r = \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\frac{u}{r} - (1+\nu)\alpha T\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}[\varepsilon_\theta + \nu\varepsilon_r(1+\nu)\alpha T] \xRightarrow{\varepsilon_r=\frac{du}{dr};\ \varepsilon_\theta=\frac{u}{r}} \sigma_\theta = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr} - (1+\nu)\alpha T\right]$$

Using equilibrium ($F_r = 0$):

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{1}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = (1+\nu)\alpha\frac{dT}{dr}$$

Double integral:

$$u = C_1 r + \frac{C_2}{r} + \frac{1+\nu}{r}\alpha\int_{r_i}^{r} Tr\,dr$$

C_1 and C_2 are constants of integration

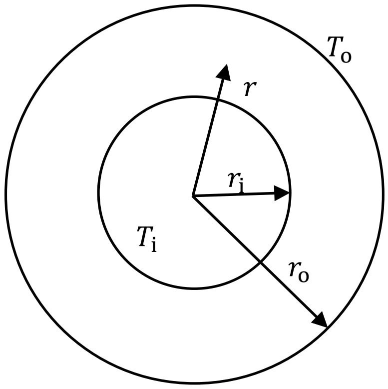
Substitute u in σ_r and σ_θ:

$$\sigma_r = A - \frac{B}{r^2} - \alpha\frac{E}{r^2}\int_{r_i}^{r} Tr\,dr$$

$$\sigma_\theta = A + \frac{B}{r^2} + \alpha\frac{E}{r^2}\int_{r_i}^{r} Tr\,dr - E\alpha T$$

where A and B are from boundary conditions

Free-free edge



B. Cs.

$@\ r = r_i \Rightarrow \sigma_r = 0$

$@\ r = r_o \Rightarrow \sigma_r = 0$

$$A = \frac{\alpha E}{r_o^2 - r_i^2}\int_{r_i}^{r_o} Tr\,dr$$

$$B = \frac{\alpha E r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} Tr\,dr$$

For solid disk:

$B = 0$

$$A = \frac{\alpha E}{r_o^2}\int_{0}^{r_o} Tr\,dr$$

Steady-state temperature disk:

$$\int_{r_i}^{r} Tr\,dr = T_1\left(\frac{r^2 - r_i^2}{2}\right) + \left(\frac{T_o - T_i}{\ln\left(\frac{r_o}{r_i}\right)}\right)\left[\frac{r^2}{2}\ln\left(\frac{r}{r_i}\right) - \left(\frac{r^2 - r_i^2}{4}\right)\right]$$

2.4.2 Plane strain

- long cylinder with ends restrained

$$\sigma_z = \nu(\sigma_r - \sigma_\theta) - E\alpha T$$

let:

$$E^* = \frac{E}{1-\nu^2};\quad \nu^* = \frac{\nu}{1-\nu};\quad \alpha^* = \alpha(1+\nu)$$

so:

$$\varepsilon_r = \frac{1}{E^*}[\sigma_r - \nu^*\sigma_\theta] + \alpha^* T$$

$$\varepsilon_\theta = \frac{1}{E^*}[\sigma_\theta - \nu^*\sigma_r] + \alpha^* T$$

Stresses due to thermal loading in a cylinder with steady state temperature distribution

$$\sigma_r = C_0\left[-\ln\left(\frac{r_0}{r}\right) - \frac{\ln k}{k^2-1}\left(1 - \frac{r_o^2}{r^2}\right)\right]$$

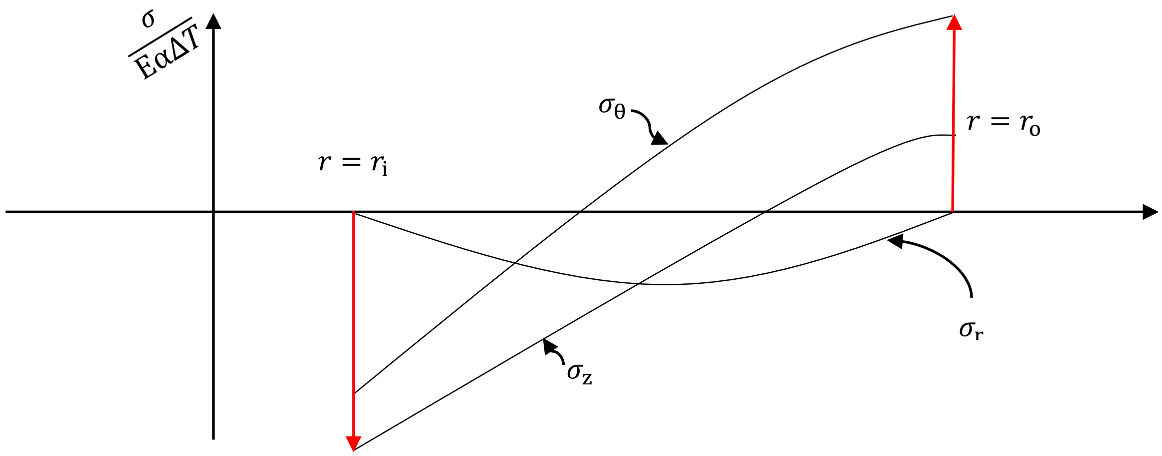
$$\sigma_\theta = C_0\left[1 - \ln\left(\frac{r_0}{r}\right) - \frac{\ln k}{k^2-1}\left(1 + \frac{r_o^2}{r^2}\right)\right]$$

where:

k is the radius ratio, $\frac{r_o}{r_i}$

$$C_0 = \frac{E'\alpha'(T_i - T_o)}{2\ln k};\; E'\alpha' \begin{cases} E\alpha \Rightarrow \text{plane stress} \\ E^*\alpha^* \Rightarrow \text{plane strain} \end{cases}$$

$$\sigma_z \Rightarrow \text{plane strain: } \sigma_z = \frac{\alpha E \Delta T}{2(1-\nu)\ln k}\left[\nu - 2\ln\left(\frac{r_o}{r}\right) - 2\nu\frac{\ln k}{k^2-1}\right]$$



- use the largest and smallest σ for failure criterion

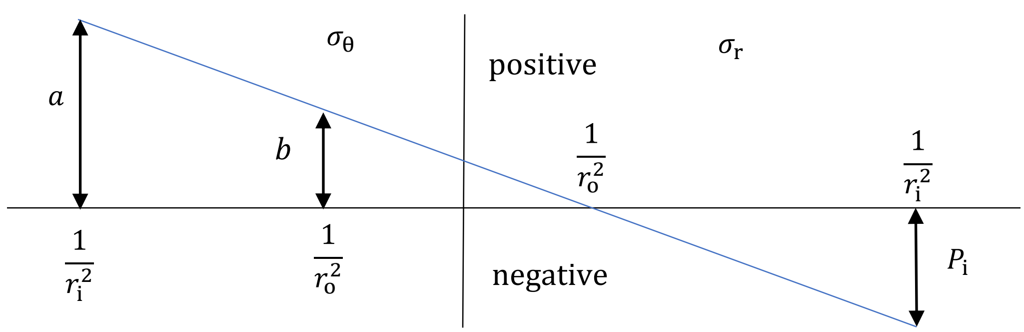
Example: A long, thick-walled steel tube with radius $r_i = 25$ mm and $r_o = 50$ mm is used to transport a hot, pressurized fluid @ $P = 10$ MPa and $T_i = 250$ °C. The outer circumference is maintained @ $T_o = 50$ °C. Find stresses at the inner and outer radius of the cylinder. Check if yielding has occurred using Tresca's criterion.

E=200 GPa, $\nu = 0.3$, $\alpha = 11 \times 10^{-6} \frac{1}{°C}$, $\sigma_y = 450$ MPa

Pressure Loads

$$\frac{1}{r_i^2} = 1600$$

$$\frac{1}{r_o^2} = 400$$



Using similar triangles:

$$\frac{a}{1600+400} = \frac{b}{400+400} = \frac{P}{1600-400}$$

$$a \Rightarrow (\boldsymbol{\sigma_\theta})_{r_i} = \mathbf{16.67\ MPa}$$

$$b \Rightarrow (\boldsymbol{\sigma_\theta})_{r_o} = \mathbf{6.67\ MPa}$$

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \Rightarrow 0.3(0\ \text{MPa} + 6.67\ \text{MPa}) \Rightarrow \boldsymbol{\sigma_z} = \mathbf{2\ MPa}$$

Thermal Loads

$$C_0 = \frac{E^*\alpha^*(T_o - T_i)}{2\ln k} \Rightarrow \frac{\left(\frac{E}{1-\nu^2}\right)\left(\alpha(1+\nu)\right)(T_o - T_i)}{2\ln k}$$

$$\Rightarrow \frac{\left(\frac{200\times 10^9\ \text{Pa}}{1-0.3^2}\right)\left(\left(11\times 10^{-6}\frac{1}{°\text{C}}\right)(1+0.3)\right)(200\ °\text{C})}{2\ln\left(\frac{50}{25}\right)} \Rightarrow \boldsymbol{C_0 = 453.42\ \text{MPa}}$$

@ $\boldsymbol{r = r_i}$:

$\Rightarrow \boldsymbol{(\sigma_r)_{r_i} = 0}$

$$\sigma_\theta = C_0\left[1 - \ln\left(\frac{r_0}{r}\right) - \frac{\ln k}{k^2-1}\left(1 + \frac{r_o^2}{r^2}\right)\right] \Rightarrow (453.42\ \text{MPa})\left[1 - \ln 2 - \frac{\ln 2}{4-1}(1+4)\right]$$

$$\Rightarrow \boldsymbol{(\sigma_\theta)_{r_i} = -389.68\ \text{MPa}}$$

$$\sigma_z = \frac{\alpha E\Delta T}{2(1-\nu)\ln k}\left[\nu - 2\ln\left(\frac{r_o}{r}\right) - 2\nu\frac{\ln k}{k^2-1}\right]$$

$$\Rightarrow \frac{\left(11\times 10^{-6}\frac{1}{°\text{C}}\right)(200\times 10^9\ \text{Pa})(200\ °\text{C})}{2(1-0.3)\ln\left(\frac{50}{25}\right)}\left[0.3 - 2\ln\left(\frac{50}{25}\right) - 2(0.3)\frac{\ln\left(\frac{50}{25}\right)}{\left(\frac{50}{25}\right)^2 - 1}\right]$$

$$\Rightarrow \boldsymbol{\sigma_z = -555.4\ \text{MPa}}$$

@ $\boldsymbol{r = r_o}$:

$\Rightarrow \boldsymbol{(\sigma_r)_{r_o} = 0}$

$$\sigma_\theta = C_0\left[1 - \ln\left(\frac{r_0}{r}\right) - \frac{\ln k}{k^2-1}\left(1 + \frac{r_o^2}{r^2}\right)\right] \Rightarrow (453.42\ \text{MPa})\left[1 - \ln 1 - \frac{\ln 2}{4-1}(1+1)\right]$$

$$\Rightarrow \boldsymbol{(\sigma_\theta)_{r_o} = -243.1\ \text{MPa}}$$

$$\Rightarrow \boldsymbol{\sigma_z = 73.2\ \text{MPa}}$$

YIELDING?

@ $\boldsymbol{r = r_i}$:

$$(\sigma_z)_t = -555.4\ \text{MPa} + 2\ \text{MPa} \Rightarrow \boldsymbol{(\sigma_z)_t = -553.4\ \text{MPa}}$$

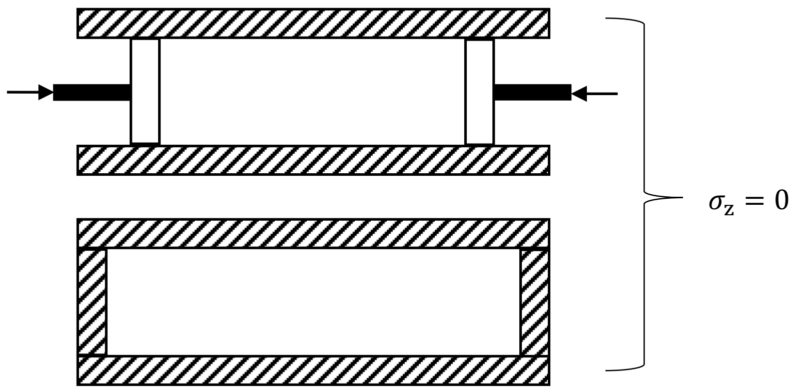
$$\text{Tresca}\begin{cases}\dfrac{\sigma_1-\sigma_2}{2}\le\dfrac{\sigma_y}{2}\\ \dfrac{\sigma_2-\sigma_3}{2}\le\dfrac{\sigma_y}{2}\\ \dfrac{\sigma_1-\sigma_3}{2}\le\dfrac{\sigma_y}{2}\end{cases}$$

$$\frac{(\sigma_z)_{\text{temp}}-(\sigma_r)_{\text{pressure}}}{2}\le\frac{\sigma_y}{2}\Rightarrow\frac{553.4-10}{2}\le\frac{450}{2}$$

$$\Rightarrow \mathbf{271.27\ MPa > 225\ MPa}$$

YIELDING OCCURS!

2.4.3 Other End Conditions due to Thermal Loads



- radial and hoop stress can be found using previous equations for plane strain
- axial stress, σ_z, from the ends

a) resultant axial force is zero

$$\int_{r_i}^{r_o}\sigma_z 2\pi r\,\mathrm{d}r = 0$$

b) $\varepsilon_z = \text{const.}\,(\text{axi}-\text{symmetric})$

$$\int_{r_i}^{r_o}\varepsilon_z 2\pi r\,\mathrm{d}r \Rightarrow \int_{r_i}^{r_o}\left\{\frac{1}{E}[\sigma_z-\nu(\sigma_r+\sigma_\theta)]+\alpha\Delta T\right\}2\pi r\,\mathrm{d}r$$

$$\varepsilon_z=\frac{2\alpha}{k^2-1}\frac{1}{r_i^2}\int_{r_i}^{r_o}Tr\,\mathrm{d}r$$

$$\sigma_z=\frac{E\alpha}{1-\nu}\left[\frac{2}{k^2-1}\frac{1}{r_i^2}\int_{r_i}^{r_o}Tr\,\mathrm{d}r-T\right]\Rightarrow\sigma_z=E\alpha\frac{T_i-T_o}{2(1-\nu)\ln k}\left[1-2\ln\left(\frac{r_o}{r}\right)-\frac{2}{k^2-1}\ln k\right]$$

3.0 Stress Functions

Direct or exact solutions to the governing equations of elasticity are not usually possible for most practical problems

<u>Airy's Stress Function</u>

In order to solve the compatibility equations for stress, Airy suggested that the elasticity be formulated in terms of one mathematical function $\varphi(x, y)$ called the stress function.

Airy's stress function $\varphi(x, y)$ is related to stress as:

$$\sigma_{\mathrm{x}} = \frac{\partial^2\varphi}{\partial y^2};\ \sigma_{\mathrm{y}} = \frac{\partial^2\varphi}{\partial x^2};\ \tau_{\mathrm{xy}} = -\frac{\partial^2\varphi}{\partial x\partial y} \tag{3.1}$$

Consider zero body forces and zero temperature change.

The compatibility equation for stress is given by:

$$\nabla^2(\sigma_{\mathrm{x}} + \sigma_{\mathrm{y}}) = 0 \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{\mathrm{x}} + \sigma_{\mathrm{y}}) = 0 \tag{3.2}$$

Substitute (3.1)→ (3.2)

$$\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} \Rightarrow \nabla^2(\nabla^2\varphi) = 0 \tag{3.3}$$

Bi-Harmonic Function

Solution (3.3) $\nabla^4\varphi = 0$ must satisfy the boundary condition.

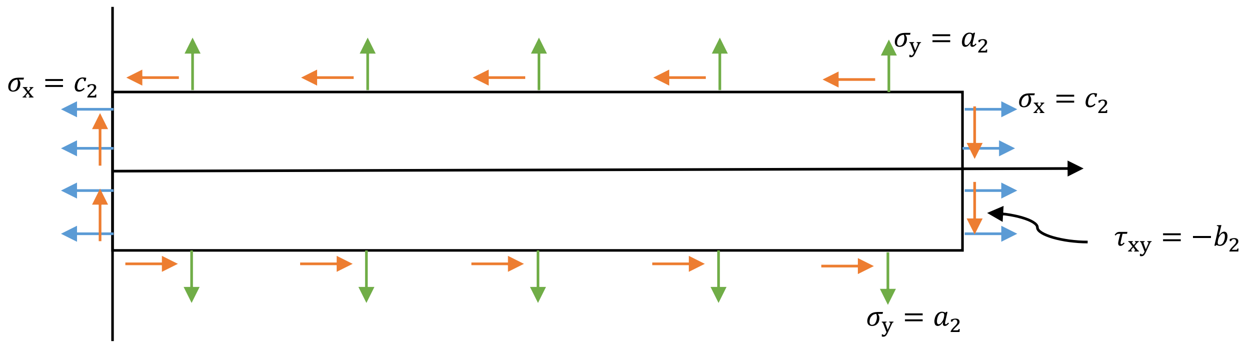
- Since (3.2) holds true for both plane stress and plain strain, so doe (3.3).
- Solving φ → stress → strain → displacement.
- For many problems, the form of the stress function is specified in advance semi-inverse method)
- In cases were specifying a stress function in advance is less obvious, the conformal mapping technique is used.
    - φ using two complex analytic functions.

3.1 Solution by Polynomial

A solution to (3.3) using polynomials are used for rectangular bodies.

a) Second degree polynomial

$$\varphi_2 = \frac{a_2}{2}x^2 + b_2 y + \frac{c_2}{2}y^2; \quad \text{where: constants} \begin{cases} \sigma_x = \dfrac{\partial^2\varphi}{\partial y^2} \Rightarrow c_2 \\ \sigma_y = \dfrac{\partial^2\varphi}{\partial x^2} \Rightarrow a_2 \\ \tau_{xy} = -\dfrac{\partial^2\varphi}{\partial x \partial y} \Rightarrow -b_2 \end{cases}$$



if $b_2 = 0$; biaxial tension

$a_2 = b_2 \Rightarrow 0$; uniaxial tension

$a_2 = c_2 \Rightarrow 0$; pure shear

b) Third order polynomial

$$\varphi_2 = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$$

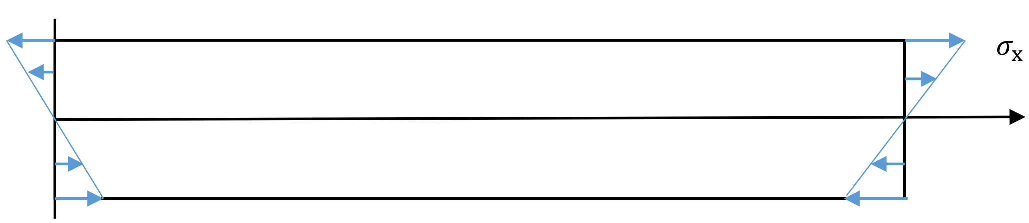
→ Satisfies (3.3)

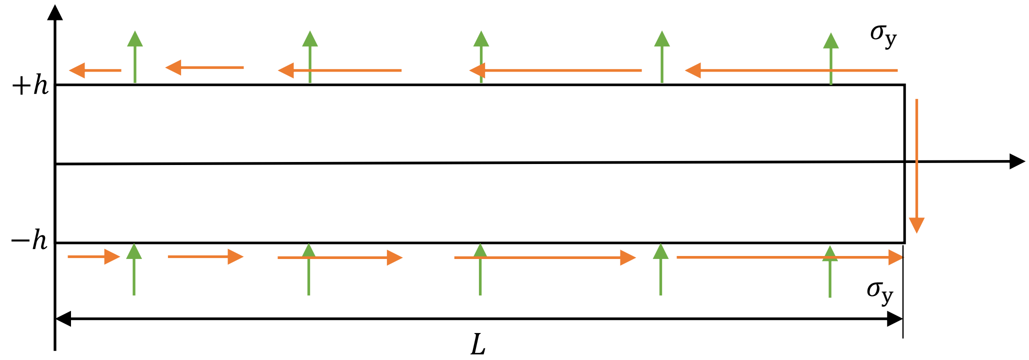
$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} \Rightarrow c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2\varphi}{\partial x^2} \Rightarrow c_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2\varphi}{\partial x \partial y} \Rightarrow -b_3 x - c_3 y$$

if $a_3 = 0;\ b_3 = 0;\ c_3 = 0$
$\sigma_x = d_3 y$
$\sigma_y = \tau_{xy} \Rightarrow 0$ (simple bending)





if $a_3 = 0;\ c_3 = 0;\ d_3 = 0$
$\sigma_y = b_3 y$
$\sigma_x = 0$
$\tau_{xy} = -b_3 x$

Method (i)

Assuming:

$$\sigma_x \propto y$$

$$\sigma_x \propto x \text{ (-BM)}$$

$$\sigma_x = c_1 xy \Rightarrow \frac{\partial^2 \varphi}{\partial y^2} \Rightarrow \frac{\partial \varphi}{\partial y} = \frac{c_1 xy^2}{2} + f_1(x)$$

$$\varphi = \frac{1}{6} c_1 xy^3 + y f_1(x) + f_2(x)$$

$$\nabla^4 \varphi = 0$$

$$y \frac{\partial^4 f_1(x)}{\partial x^4} + \frac{\partial^4 f_2(x)}{\partial x^4} = 0$$

$$y \frac{\partial^4 f_1(x)}{\partial x^4} = 0 \quad ;\ f_1(x) = c_2 x^3 + c_3 x^2 + c_4 x + c_5$$
$$\frac{\partial^4 f_2(x)}{\partial x^4} = 0 \quad ;\ f_2(x) = c_6 x^3 + c_7 x^2 + c_8 x + c_9$$

$$\varphi = \frac{1}{6} c_1 xy^3 + y(c_2 x^3 + c_3 x^2 + c_4 x + c_5) + c_6 x^3 + c_7 x^2 + c_8 x + c_9$$

$$\sigma_y = \frac{\partial^2 \varphi}{\partial x^2} \Rightarrow 6[(2y + c_6)x + 2(c_3 y + c_4)]$$

$$\tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \Rightarrow -\frac{1}{2} c_1 y^2 - 3c_2 x^2 - 2c_3 x - c_4$$

<u>BCs</u>

$$\text{for all } x \begin{cases} \sigma_{\text{y}} = 0 \ @ \pm\ h \\ \tau_{\text{xy}} = 0 \ @ \pm\ h \end{cases}$$

$c_2, c_3, c_6,$ and c_7 are all zero

$$c_4 = -\frac{1}{2} c_1 h^2$$

$$\tau_{\text{xy}} = -\frac{1}{2} c_1 (y^2 - h^2)$$

c) 4th order polynomial

$$\varphi_4 = \frac{a_4}{(4)(3)} x^4 + \frac{b_4}{(3)(2)} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{(3)(2)} x y^3 + \frac{e_4}{(4)(5)} y^4 \tag{3.8}$$

$e_4 = -(2c_4 + a_4) \Rightarrow$ satisfies equation (3.3)

$$\begin{gathered} \sigma_{\text{x}} = \frac{\partial^2 \varphi_4}{\partial y^2} \Rightarrow \frac{b_4}{6} x^3 + \frac{c_4}{2} x^2 y + 3\frac{d_4}{6} y^2 + 4\frac{e_4}{(4)(3)} y^3 \\ \Rightarrow c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2 \\ \sigma_{\text{y}} = \frac{\partial^2 \varphi_4}{\partial x^2} \Rightarrow a_4 x^2 + b_4 xy + c_4 y^2 \\ \tau_{\text{xy}} = -\frac{\partial^2 \varphi}{\partial x \partial y} \Rightarrow -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2 \end{gathered} \tag{3.9}$$

Take all of the coefficients except d_4 to be zero.

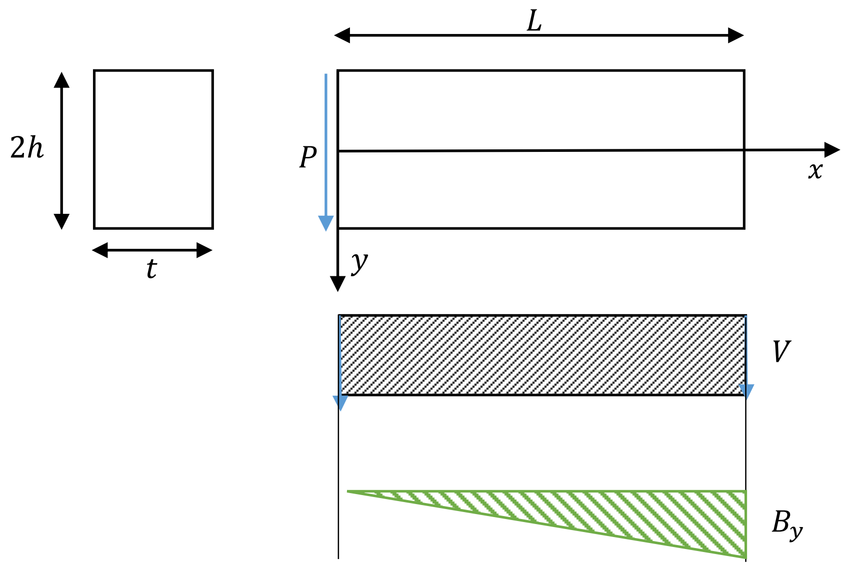
$$\sigma_{\text{x}} = d_4 xy, \quad \sigma_{\text{y}} = 0, \quad \tau_{\text{xy}} = -\frac{d_4}{2} y^2$$

$\Rightarrow \sigma_{\text{x}}$ varies linearly in x and y direction

$\Rightarrow @\ y = \pm h, \tau_{\text{xy}}$ is uniform

$\Rightarrow$ along the transverse direction, $\tau_{\text{xy}} \propto y^2$

Example



$$P = \int_{-h}^{h} \tau_{\mathrm{xy}} t \, \mathrm{d}y + \int_{-h}^{h} \frac{1}{2} c_1 t (y^2 - h^2) \, \mathrm{d}y$$

$$c_1 = -\frac{3P}{2th^3} \Rightarrow \frac{P}{I}$$

$$\sigma_{\mathrm{x}} = c_1 xy \Rightarrow \left(\frac{-P}{I}\right) xy = \frac{My}{I}$$

$$\tau_{\mathrm{xy}} = \left(\frac{-P}{2I}\right)(h^2 - y^2)$$

Method (iii)

Superposition: φ_2 and φ_4 with

$a_2, c_2, a_4, b_4, c_4,$ and e_4 all equal zero

$$\varphi = \varphi_2 + \varphi_4$$

$$\varphi = b_2 xy + \frac{d_4}{6} xy^3$$

$$\sigma_{\mathrm{x}} = \frac{\partial^2 \varphi}{\partial y^2} \Rightarrow d_4 xy$$

$$\sigma_{\mathrm{y}} = \frac{\partial^2 \varphi}{\partial x^2} \Rightarrow 0$$

$$\tau_{\mathrm{xy}} = -b_2 - \frac{d_4}{2} y^2$$

$$\tau_{\mathrm{xy}} = 0 \ @ \pm h$$

$$d_4 = -\frac{2b_2}{h_2}$$

$$\tau_{\mathrm{xy}} = -b_2\left(1 - {y^2}/{h^2}\right)$$

$$P = \int_{-h}^{h} \tau_{\mathrm{xy}} t \,\mathrm{d}y \Rightarrow \int_{-h}^{h} b_2 t\left(1 - {y^2}/{h^2}\right)\mathrm{d}y \Rightarrow b_2 = -\frac{3P}{4ht} \Rightarrow \frac{Ph^2}{2\mathrm{I}}$$

$$\sigma_{\mathrm{x}} = -\frac{Pxy}{\mathrm{I}}$$

$$\tau_{\mathrm{xy}} = -\frac{P}{2\mathrm{I}}(h^2 - y^2)$$

3.2 Stress Concentrations

Stress concentrations refer to stresses which have higher than normal stresses

- Yielding, fatigue, fracture, etc.

These arise from loading discontinuities or geometrical discontinuities.

<u>Loading Discontinuity</u>

- Loading through a knife edge, point, ball bearing or roller bearings, or gear teeth

<u>Geometry Discontinuity</u>

- holes, notches, change in cross-section
- more pronounced in brittle material than in ductile
    - find stress concentrations using analytical, numerical, or experimental methods
- often, it is more convenient to use polar coordinates

3.2.1 Use of Polar Coordinates

$$\nabla^4\varphi = \nabla^2/\nabla^2\varphi \Rightarrow 0$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \text{ rectangular coordinates}$$

$$\nabla^2 = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \tag{3.12}$$

$$\sigma_{\mathrm{r}} = (\sigma_{\mathrm{x}})_{\theta=0} \Rightarrow \frac{\partial^2\varphi}{\partial y^2} = \left(\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right) \tag{3.13}$$

$$\sigma_\theta = (\sigma_y)_{\theta=0} \Rightarrow \frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial r^2}$$

$$\tau_{r\theta} = (\tau_{xy})_{\theta=0} \Rightarrow \left(-\frac{\partial^2\varphi}{\partial x \partial y}\right)_{\theta=0} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

For axis symmetry, φ is independent of angle θ and equation (3.3) reduces to:

$$\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right) \Rightarrow 0 \quad \text{Euler Equation} \tag{3.14}$$

Equation (3.13) can be reduced to:

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r};\ \sigma_\theta = \frac{\partial^2\varphi}{\partial r^2};\quad \tau_{r\theta} = 0 \tag{3.15}$$

General solution of (3.14)

$$\sigma_r = c_1 \ln r + c_2 r^2 \ln r + c_3 r^2 + c_4 \tag{3.16}$$

where c_1 to c_4 are constants

$$\begin{aligned} \sigma_r &= \frac{c_1}{r_2} + c_2(1 + 2\ln r) + 2c_3 \\ \sigma_\theta &= -\frac{c_1}{r^2} + c_2(3 + 2\ln x) + 2c_3 \\ \tau_{r\theta} &= 0 \end{aligned} \tag{3.17}$$

If there are no holes, c_1 and c_2 are zero.

3.2.2 Stresses due to Concentration Loads

- point load (mathematic abstraction)
- in practice, "point" loads are always applied at a small area
- analytical solutions therefore are NOT valid at the point

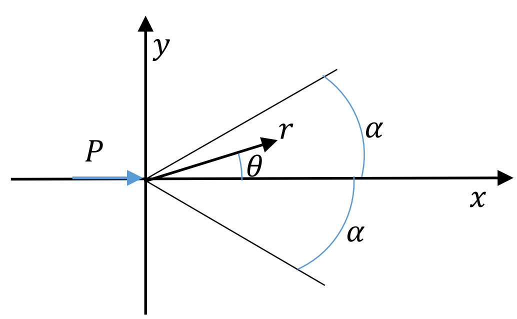
<u>Example</u>

Comparison of a wedge (unit thickness)

→ indenter

$$\varphi = cPr\theta \sin\theta$$

- satisfies (3.3)



$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \Rightarrow \frac{1}{r}2cP\cos\theta$$

$$\sigma_\theta = \frac{\partial^2 \varphi}{\partial r^2} \Rightarrow 0$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \varphi}{\partial \theta \partial r} = 0$$

$$\left.\begin{array}{l} P - 2\int_0^\alpha \sigma_r \cos\theta \; r d\theta = 0 \Rightarrow P = -2\int_0^\alpha 2cP\frac{\cos^2\theta}{r} \; r d\theta \\ \cos^2\theta = \dfrac{\cos 2\theta + 1}{2} \end{array}\right\} \begin{array}{l} 4c\int_0^\alpha \dfrac{\cos 2\theta + 1}{2} \; d\theta = -1 \\ c = -\dfrac{1}{2\alpha + \sin 2\alpha} \end{array}$$

$$\sigma_r = \frac{-2P}{2\alpha + \sin 2\alpha}\frac{\cos\theta}{r} \quad (3.19)$$

$$\sigma_\theta = 0, \;\; \tau_{r\theta} = 0$$

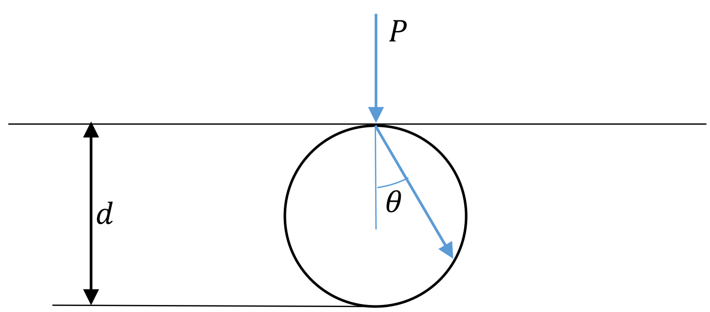
Special Case where $\alpha = \frac{\pi}{2}$

Flamant Problem

- point or line load on semi-infinite solid

$$\sigma_r = \frac{-2P}{\pi}\frac{\cos\theta}{r} \quad (3.20)$$

$$\sigma_\theta = 0, \;\; \tau_{r\theta} = 0$$

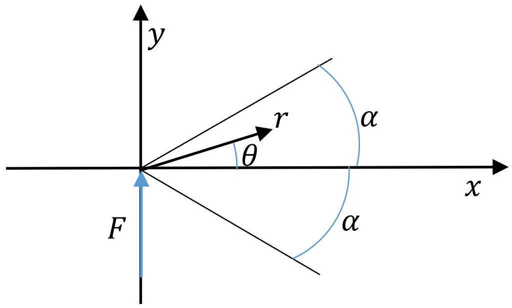


$r = d\cos\theta$

$$\sigma_r = \frac{-2P}{\pi d} \quad (3.21)$$

constant for a given d at all points on the circle

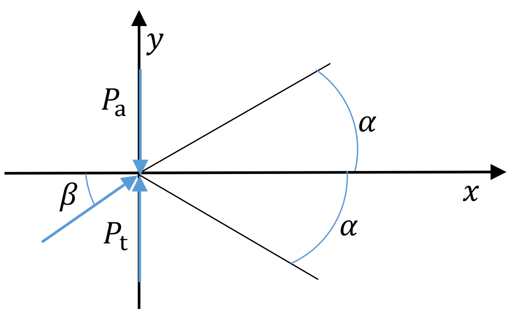
(b) Bending of a wedge:



$$\varphi = aFr\left(\frac{\pi}{2} - \theta\right)\sin\left(\frac{\pi}{2} - \theta\right) \Rightarrow aFr\theta_1 \sin\theta_1$$

$$\sigma_r = \frac{-2F\cos\theta_1}{r(2\alpha - \sin 2\alpha)} \Rightarrow \frac{-2F\cos\theta_1}{r(2\alpha - \sin 2\alpha)}$$

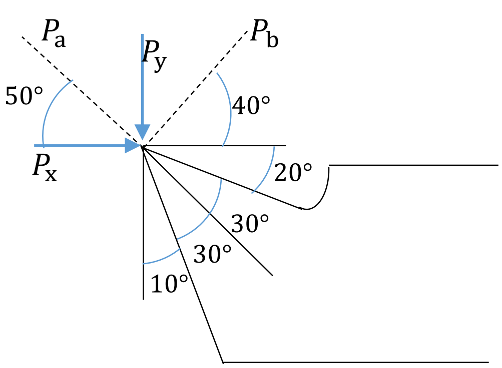
$\sigma_\theta = \tau_{r\theta} \Rightarrow 0$

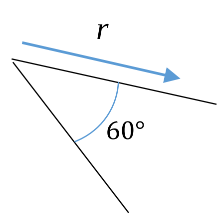


$$\sigma_r = -\frac{2}{tr}\left[\frac{P_a\cos\theta}{2\alpha + \sin 2\alpha} + \frac{P_t\sin\theta}{2\alpha - \sin 2\alpha}\right]$$

Example:

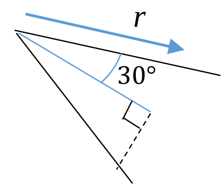
An orthogonal cutting tool has a rake angle of 20°, clearance angle of 10°, and width of cut of 4 mm. The horizontal and vertical loads are $P_x = 1.5$ kN, $P_y = 2.5$ kN. Find the minimum tool nose radius of $\sigma_r \leq 900$ MPa.





Mitchell's Solution

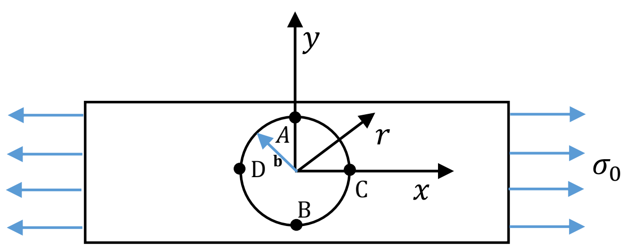
$$\sigma_r = -\frac{2}{(\omega)r}\left[\frac{P_a \cos\theta}{2\alpha + \sin 2\alpha} + \frac{P_b \sin\theta}{2\alpha - \sin 2\alpha}\right]$$
$$P_a = P_x \cos 50° + P_y \cos 40° \Rightarrow 5.75 \text{ kN}$$
$$P_b = P_y \sin 40° + P_x \sin 50° \Rightarrow 0.916 \text{ kN}$$
$$\alpha = \theta \Rightarrow 30° = \frac{\pi}{6}$$
$$\Rightarrow -\frac{2}{4r}\left[\frac{(5.75 \times 10^3)\cos 30°}{\frac{\pi}{3} + \sin 60°} + \frac{0.916 \times 10^3)\sin 30°}{\frac{\pi}{3} - \sin 60°}\right]$$
$$= 900 \text{ MPa}$$



$$\boldsymbol{r = 2.85\ \text{mm}};\ r_t = r\tan 30° \Rightarrow \boldsymbol{r_t = 1.65\ \text{mm}}$$

3.2.3 Stress Concentrations around a Small Circular Hole (Kirsh Solution)

$$\varphi(r,\theta) = \frac{\sigma_\theta}{2}\left(\frac{r^2}{2} - b^2 \ln r\right) + \frac{\sigma_0}{4}\left(2b^2 - r^2 - \frac{b^4}{r^2}\right)\cos 2\theta$$



$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} \Rightarrow \frac{\sigma_0}{2}\left(1 - \frac{b^2}{r^2}\right) + \frac{\sigma_0}{2}\left(1 - 4\frac{b^2}{r^2} + 3\frac{b^4}{r^4}\right)\cos 2\theta$$
$$\sigma_\theta = \frac{\partial^2\varphi}{\partial r^2} \Rightarrow \frac{\sigma_0}{2}\left(1 + \frac{b^2}{r^2}\right) + \frac{\sigma_0}{2}\left(-1 - 3\frac{b^4}{r^4}\right)\cos 2\theta \qquad (3.23)$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \Rightarrow \frac{\sigma_0}{2}\left(-1 - 2\frac{b^2}{r^2} + 3\frac{b^4}{r^4}\right)\sin 2\theta$$

Points A and B have maximum stress:

$$\frac{\pi}{2}, \frac{3\pi}{2}$$

C: 0 rad

D: π rad

Putting $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ into equation (3.23) for σ_θ at $b = r$,

$$\sigma_\theta = 3\sigma_0 \Rightarrow \sigma_x$$

Stress concentration: $\frac{3\sigma_0}{\sigma_0} = 3$

$$@\ \theta = 0 \text{ and } \pi:$$

$$\sigma_\theta = \sigma_y \Rightarrow -\sigma_0 \text{ (compressive)}$$

$$@\ r > 10b$$

$$(\sigma_\theta)_{\frac{\pi}{2},\ \frac{3\pi}{2}} = \frac{\sigma_0}{2}\left(2 + \frac{b}{r^2}^2 + 3\frac{b}{r^4}^4\right) \Rightarrow \sigma_0$$

Result is valid for finite width plate $w > 10b$.

4.0 Shock and Impact Loading

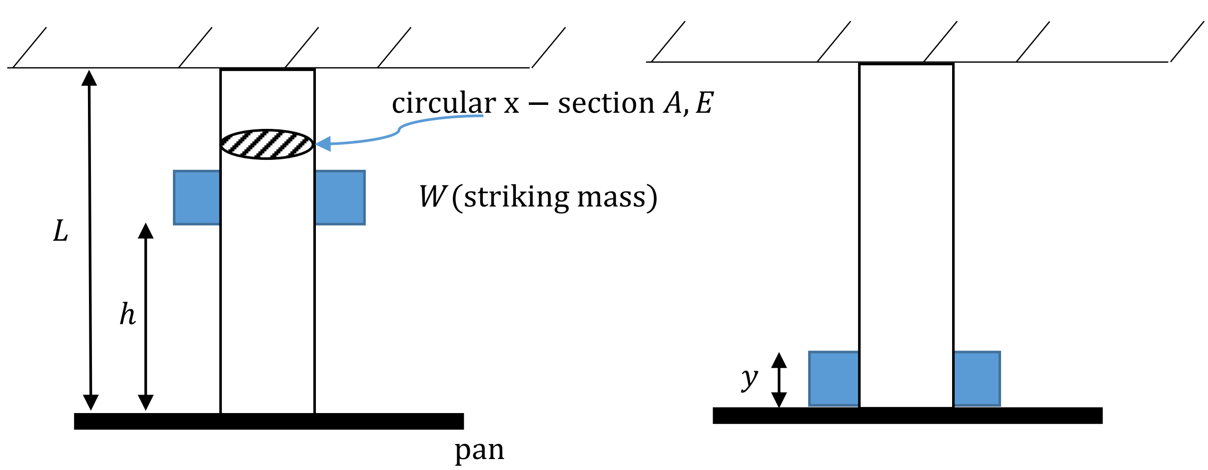
- When forces or displacements are applied suddenly to structural members, the stress levels and deformation are often much higher than those that occur by the same forces or displacements applied gradually
- Rapidly moving loads: (train crossing a bridge, drop of a forge hammer, sudden loads produced during combustion in the power stroke of an internal combustion engine, collision, acceleration, etc.)
- Impact shock load when

$$\left.\begin{matrix} t_{\text{applied}} \leq \frac{1}{2}T \\ t_{\text{applied}} \leq T \end{matrix}\right\} \text{time of highest natural frequency of the structure}$$

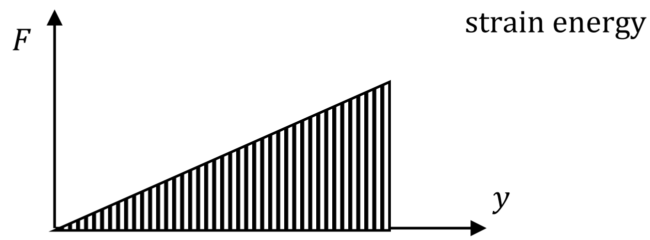
- Quasi-static:

$$t_{\text{applied}} \geq 3T$$

4.1 Stress and Displacement in an Elastic Axial Bar



- mass of the pan and the bar is small compared to the striking mass (W)
- $y \neq f(t)$
- Linear elastic
- NO energy lost



Energy balance:

external work = stored strain energy of the bar

$$W(h+\hat{y}) = \frac{1}{2}\hat{F}\hat{y}; \quad (\hat{\ } \text{ denotes maximum})$$

$$\hat{\sigma} = E\hat{\varepsilon} \Rightarrow E\frac{\hat{y}}{L} = \hat{y} \Rightarrow \frac{\hat{\sigma}L}{E}$$

$$\hat{F} = \hat{\sigma}A$$

$$W\left(h + \frac{\hat{\sigma}L}{E}\right) = \frac{1}{2}\hat{\sigma}A\frac{\hat{\sigma}L}{E}$$

$$\frac{AL}{2E}\hat{\sigma}^2 - \left(\frac{WL}{E}\right)\hat{\sigma} - Wh = 0 \tag{4.1}$$

$$\hat{\sigma} = \frac{W}{A}\underbrace{\left[1 + \sqrt{1 + \frac{2hEA}{WL}}\right]}_{\Delta \Rightarrow \text{ impact factor}} \tag{4.2}$$

$$\hat{y} = \frac{WL}{AE}\left[1 + \sqrt{1 + \frac{2hEA}{WL}}\right] \tag{4.3}$$

$$\hat{F} = W\left[1 + \sqrt{1 + \frac{2hEA}{WL}}\right] \tag{4.4}$$

$\hat{F}$ is the effective dynamic fore due to impact.

$$y_{\text{static}} = \frac{WL}{AE}$$

Impact, Δ

$$\Delta = 1 + \sqrt{1 + \frac{2hEA}{WL}} \tag{4.5}$$

$$\Delta = 1 + \sqrt{1 + \frac{2h}{y_{\text{static}}}}$$

- If the load is applied suddenly even at h=0;

$$\hat{\sigma} = \frac{2W}{A} \Rightarrow 2(\sigma_{\text{max}})_{\text{static}}$$

$$\hat{F} = 2(F_{\text{max}})_{\text{static}}$$

$$\hat{y} = 2(y_{\text{max}})_{\text{static}}$$

- $\hat{\sigma}$ can be reduced by increasing L or A and decreasing

$$\sigma_{\text{static}} = \frac{F}{A}$$

Applications:

- wood is used as railway track
- long bolts used to attach ends of the pneumatic cylinder of jack hammers
- if the bar has a weight of q per unit length, the impact factor, Δ can be modified to:

$$\Delta^* = \left[1 + \sqrt{1 + \frac{2hEA}{WL}\left(\frac{1}{1 + \frac{qL}{3W}}\right)}\right] \tag{4.7}$$

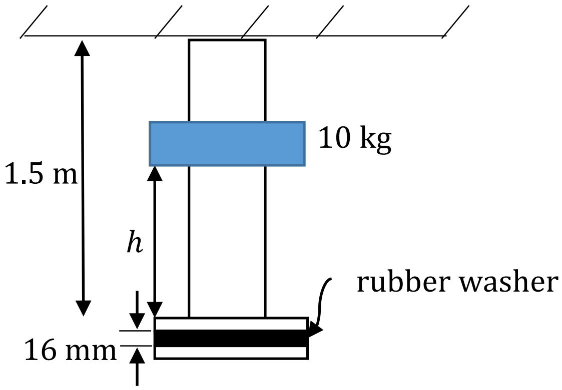
- Analysis is similar for cases involving impact of moving loads.

$$(\text{KE})_{\text{moving load}} = u_{\text{structure}}$$

<u>Example (a)</u>

Find the max stress in a steel rod.

(i) with washer
(ii) without washer



rubber washer:

$$k = 9\,\frac{\mathrm{N}}{\mathrm{mm}}$$
$$\emptyset = 15\ \mathrm{mm}$$
$$E = 200\ \mathrm{GPa}$$

i) with washer

$$\hat{\sigma} = \frac{W}{A}\left[1 + \sqrt{1 + \frac{2h}{y_{\mathrm{static}}}}\right]$$

$$W = (10\ \mathrm{kg})\left(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}\right) \Rightarrow 98.1\ \mathrm{N}$$

$$A = \frac{\pi}{4}\emptyset^2 \Rightarrow \frac{\pi}{4}(15\ \mathrm{mm})^2 = 177\ \mathrm{mm}^2$$

$$y_{\mathrm{static}} = \underbrace{\frac{WL}{AE}}_{\text{without washer}} + \underbrace{\frac{W}{k}}_{\text{with washer}} \Rightarrow \frac{(98.1\ \mathrm{N})(1\,500\ \mathrm{mm})}{(177\ \mathrm{mm}^2)\left(200\times 10^3\,\frac{\mathrm{N}}{\mathrm{mm}^2}\right)} + \frac{(98.1\ \mathrm{N})}{9\,\frac{\mathrm{N}}{\mathrm{mm}}} \Rightarrow \boldsymbol{y_{\mathrm{static}} = 10.9\ \mathrm{mm}}$$

without washer with washer

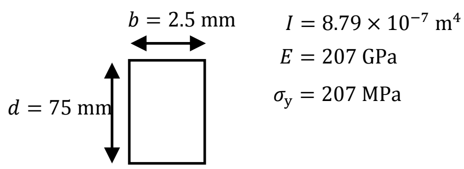
$$\hat{\sigma} = \frac{98.1\ \mathrm{N}}{177\ \mathrm{mm}^2}\left[1 + \sqrt{1 + \frac{2(1\,000\ \mathrm{mm})}{10.9\ \mathrm{mm}}}\right] \Rightarrow \boldsymbol{\hat{\sigma} = 8.08\ \mathrm{MPa}}$$

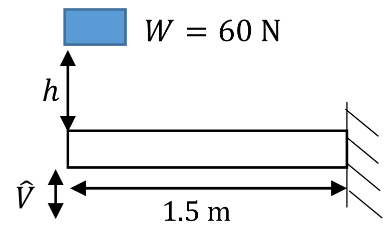
ii) without washer

$$\hat{\sigma} = \frac{98.1\ \mathrm{N}}{177\ \mathrm{mm}^2}\left[1 + \sqrt{1 + \frac{2(1\,000\ \mathrm{mm})}{4.157\times 10^{-3}\ \mathrm{mm}}}\right] \Rightarrow \boldsymbol{\hat{\sigma} = 385\ \mathrm{MPa}}$$

<u>Example (b)</u>

Find the max h allowed to avoid yielding.





Equating external energy = stored strain energy:

$$W(h+\hat{V}); \quad \hat{V} = \frac{PL^3}{3EI} \Rightarrow P = \text{dynamic load}$$

$$\hat{\sigma} = \frac{My}{I} \Rightarrow \hat{\sigma} = \frac{PL\,{}^{d}\!/_{2}}{I} \Rightarrow P = \frac{\hat{\sigma} I}{L\,{}^{d}\!/_{2}}$$

$$\hat{V} = \left(\frac{\hat{\sigma} I}{L\,{}^{d}\!/_{2}}\right)\frac{L^3}{3EI}$$

$$PE = W\left(h + \left(\frac{\hat{\sigma} I}{L\,{}^{d}\!/_{2}}\right)\frac{L^3}{3EI}\right)$$

Stored strain energy:

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,\mathrm{d}x \Rightarrow U = \frac{1}{2}\int_0^L \frac{(Px)^2}{EI}\,\mathrm{d}x \Rightarrow \frac{P^2L^2}{6EI} \Rightarrow U = \left(\frac{\hat{\sigma} I}{L\,{}^{d}\!/_{2}}\right)^2 \frac{L^3}{6EI}$$

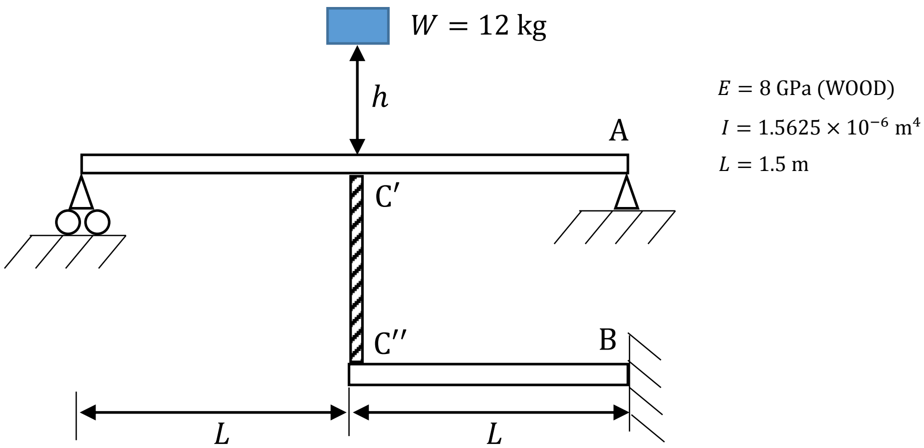
$PE = U$

$$W\left(h + \frac{\hat{\sigma} L^2}{3\left({}^{d}\!/_{2}\right)E}\right) = \frac{\hat{\sigma}^2 IL}{6\left({}^{d}\!/_{2}\right)^2 E}$$

$$\Rightarrow 60\left(h + \frac{(270 \times 10^6\ \text{Pa})(1.5\ \text{m})^2}{3(0.0375\ \text{m})(207 \times 10^9\ \text{Pa})}\right) = \frac{(270 \times 10^6\ \text{Pa})^2(8.79 \times 10^{-7}\ \text{m}^4)(1.5\ \text{m})}{6(0.0375\ \text{m})^2(207 \times 10^9\ \text{Pa})}$$

$$\boldsymbol{h = 0.891\ \text{m}}$$

Example (c)

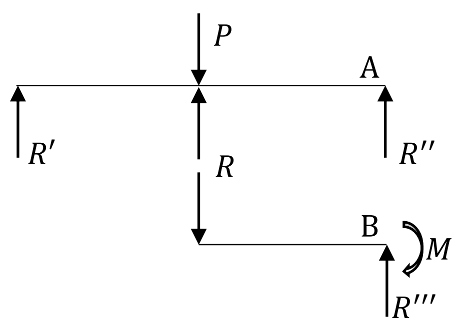


(i) max h allowed if max deflection is limited to 38 mm
(ii) amount of energy absorbed by each beam

$PE = U$

$PE = W(h + \delta_{\text{max}})$

FBD



(i)

Beam B

$$\delta_B = \frac{RL^3}{3EI} \Rightarrow R = \frac{\delta_B 3EI}{L^3} \Rightarrow R = \frac{(0.038 \text{ m})3(8 \times 10^9 \text{ Pa})(1.5625 \times 10^{-6} \text{ m}^4)}{(1.5 \text{ m})^3} \Rightarrow \boldsymbol{R = 422.2 \text{ N}}$$

Beam A

$$\delta_{\mathrm{A}} = \frac{W'L^3}{48EI} \Rightarrow \frac{(P-R)(2L)^3}{48EI} \Rightarrow P = \frac{\delta_{\mathrm{A}}6EI}{L^3} + R$$

$$\Rightarrow P = \frac{(0.038\ \mathrm{m})6(8\times10^9\ \mathrm{Pa})(1.5625\times10^{-6}\ \mathrm{m}^4)}{(1.5\ \mathrm{m})^3} + 422.2\ \mathrm{N}$$

$$\boldsymbol{P = 1\,266.7\ \mathrm{N}}$$

$$\Delta = 1 + \sqrt{1 + \frac{2h}{y_{\mathrm{static}}}} \Rightarrow \Delta = \frac{P}{W} \Rightarrow \frac{1\,266.7\ \mathrm{N}}{12\ \mathrm{kg}\ (9.81\frac{\mathrm{m}}{\mathrm{s}^2})} \Rightarrow \Delta = \mathbf{10.76}$$

$$\boldsymbol{h = 166.4\ \mathrm{mm}}$$

$$\delta_{\mathrm{static}} = \frac{\delta_{\mathrm{max}}}{\Delta} \Rightarrow \frac{0.038\ \mathrm{m}}{10.76}$$

$PE = U$

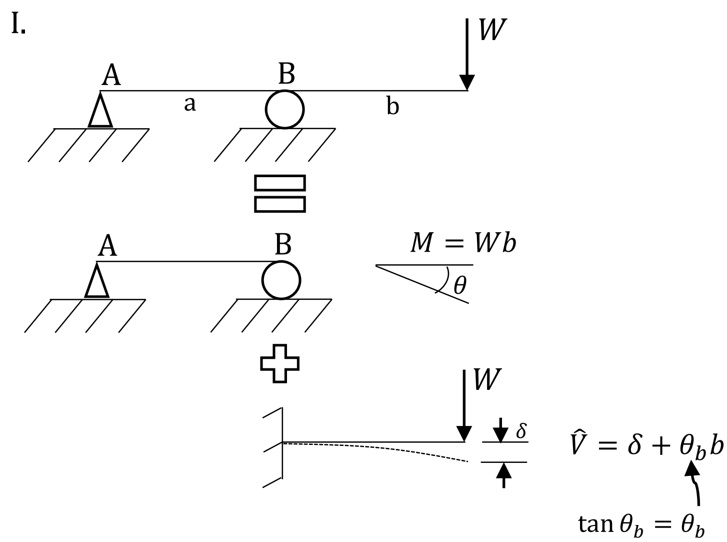
$$W(h + \delta_{\mathrm{max}}) = \frac{1}{2}P\delta; \quad \boldsymbol{h = 166.4\ \mathrm{mm}}$$

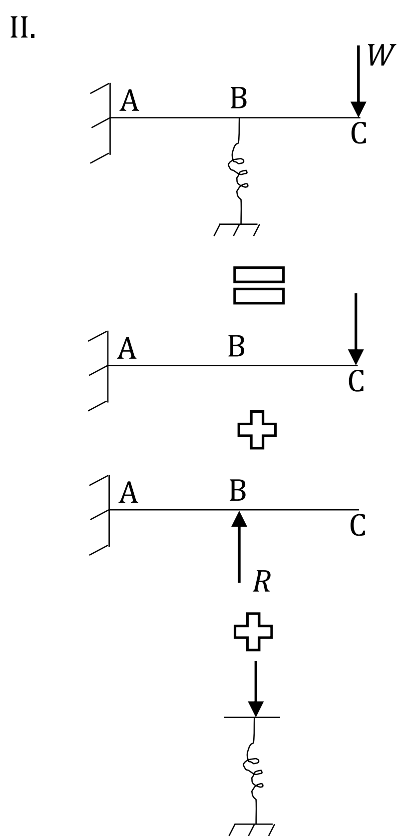
(ii)

$$U_{\mathrm{A}} = \frac{1}{2}P'\delta' \Rightarrow \frac{1}{2}(P-R)\delta \Rightarrow \boldsymbol{U_{\mathrm{A}} = 16.05\ \mathrm{J}}$$

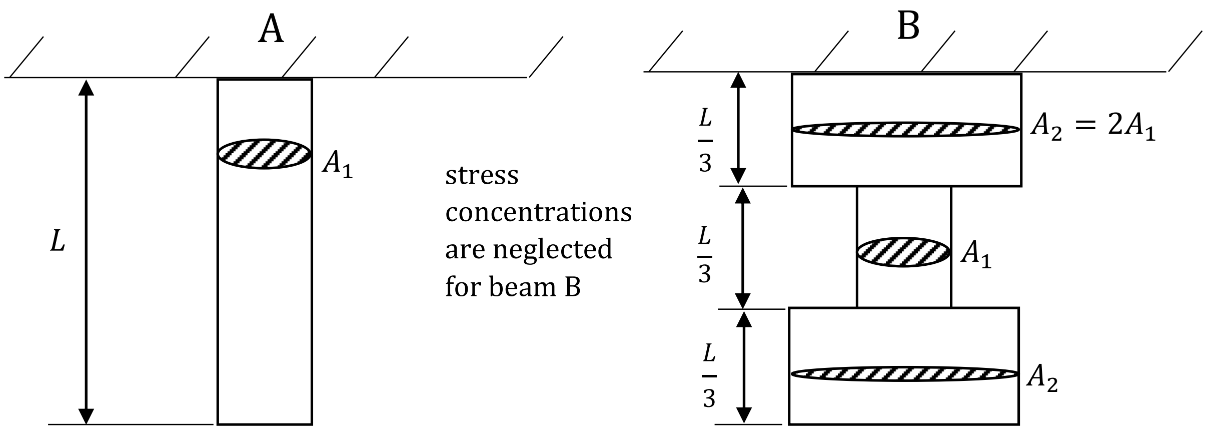
$$U_{\mathrm{B}} = \frac{1}{2}P'\delta' \Rightarrow \frac{1}{2}R\delta \Rightarrow \boldsymbol{U_{\mathrm{A}} = 8.02\ \mathrm{J}}$$

Superposition can be used to obtain the deflection.





4.2 Effects of Geometry



$$U_{\mathrm{A}} = \frac{1}{2}\hat{P}\delta$$

$$\hat{P} = \hat{\sigma}A, \delta = \varepsilon L, \hat{\sigma} = E\varepsilon$$

$$U = \frac{1}{2}\frac{\hat{\sigma}^2}{E}\ AL \qquad\qquad U_{\mathrm{B}} = \frac{1}{2}\frac{\hat{\sigma}^2}{E}\ A_1\frac{L}{3} + 2\left[\frac{1}{2}\frac{\hat{\sigma}^2}{E}\ A_2\frac{L}{3}\right]$$

$$U_{\mathrm{A}} = \frac{1}{2}\frac{\hat{\sigma}^2}{E}\ A_1 L \qquad\qquad U_{\mathrm{B}} = \frac{2}{3}U_A$$

($A_2 = 2A_1$)

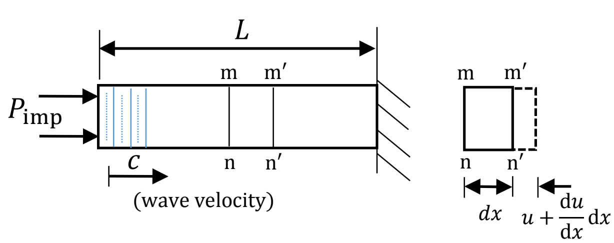
To design or maximum resistance to impact load:

- use a low E
- increase length, L
- distribute the stress as uniformly as possible
- avoid stress concentrations

4.3 Stress Wave Propagation under Impact Loading

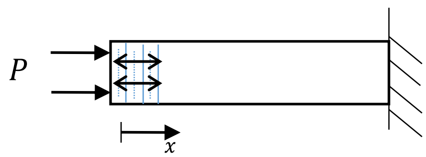
The above provides a quick solution estimate of maximum stress and deflection under impact loading. Exact consideration involves the motion of stress waves traveling within the structures.

Consider the 1-D problem:

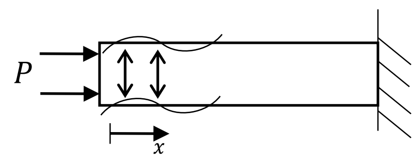


At the instant of impact of the free end, a very thin segment of the bar directly under the load is set into motion. The rest of the bar, remote from the load, P, remains undisturbed for a small, finite length of time. The deformation due to P propagates along the bar with time in a form of an elastic deformation: stress wave. If L is large, time required for the wave propagation becomes significant and must be considered.

Two different types of stress waves:



Longitudinal Wave
Particle motion in the x-direction
Pressure wave (P_{wave})
Velocity (c_{L})



Transverse Wave
Shear wave (s-wave)
Particle motion in the y-direction

$$F = ma \qquad m = \rho V$$
$$F = \sigma A \qquad m = \rho A\, dx$$
$$\sigma = E\varepsilon \qquad a = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$
$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \qquad EA\frac{\mathrm{d}u}{\mathrm{d}x} = \rho A\, dx \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$
$$F = EA\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{E}{\rho}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$

$$\boldsymbol{c^2 = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}}$$

1-D wave equation

$c_{\text{L}} = \sqrt{\frac{E}{\rho}}$: velocity of longitudinal wave

$c_{\text{S}} = \sqrt{\frac{G}{\rho}}$: velocity of transverse wave

e.g.

steel: $c_{\text{L}} = \sqrt{\frac{207\times10^9}{7\,850}} \Rightarrow 5\,135\,\frac{\text{m}}{\text{s}}$

copper: $c_{\text{s}} = \sqrt{\frac{120\times10^9}{8\,900}} \Rightarrow 3\,672\,\frac{\text{m}}{\text{s}}$

Verify that $f(x \pm ct)$ is a solution of equation (4.8)

where;

$+$ is the negative x direction
$-$ is the positive x direction

$$u(x,t) = f_1(x - ct) + f_2(x + ct)$$

$$u = A \sin\left(\frac{2\pi}{L}\right)(x \mp ct)$$

$$\boldsymbol{u_i} = \boldsymbol{A_i e^{i(\omega t - k_i x_i)}}$$

k: wave number $\frac{2\pi}{\lambda_{\text{wave}}}$

ω: angular frequency

Stress Waves at Fixed and Free Ends

<u>@ fixed end</u>

Waves are reflected and unchanged i.e. compressive stays compressive

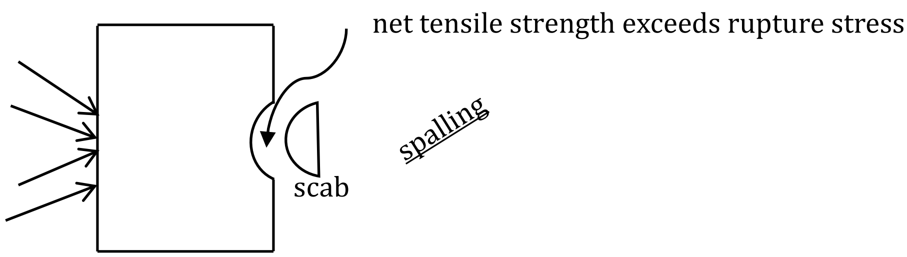
<u>@ free end</u>

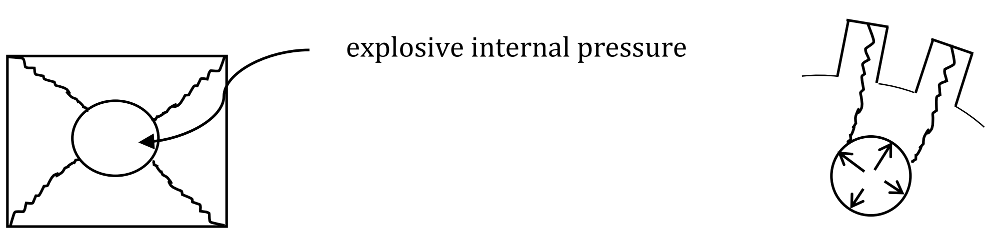
Waves are reflected with same amplitude but opposite in sign.

compressive → tensile

tensile → compressive

2-D (longitudinal and shear)



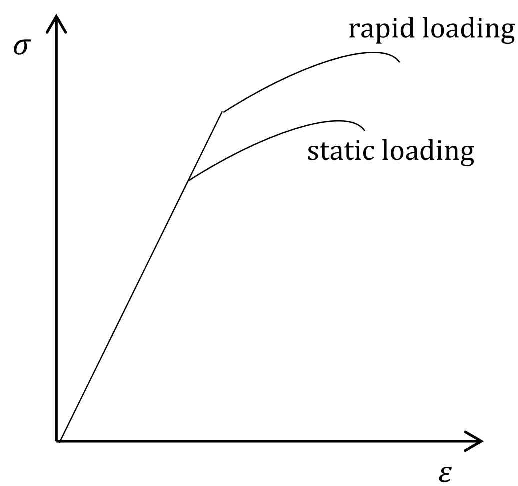


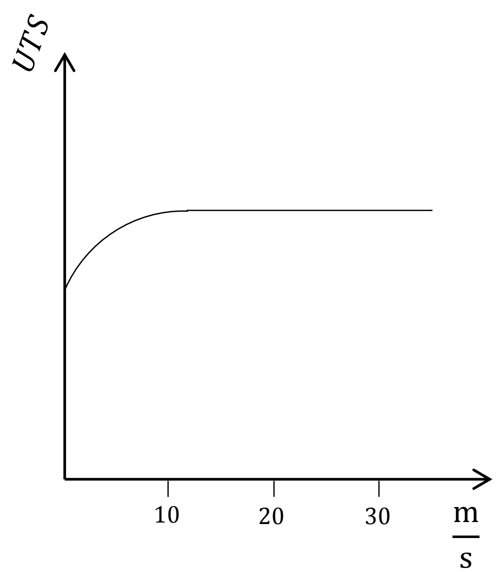
Fracture occurs at the thickest section as compared to thin section in static loading.

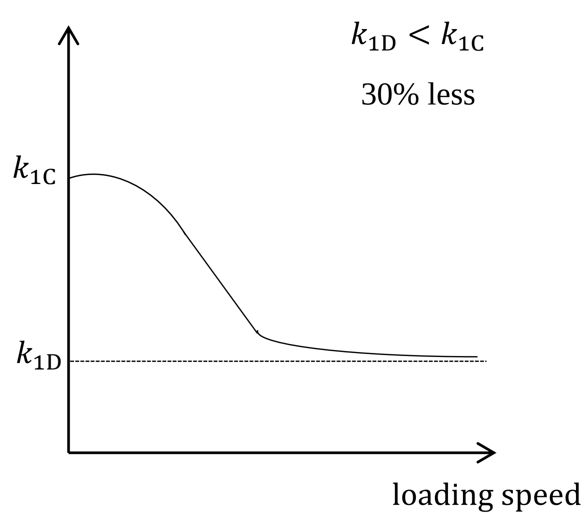
In real materials, there is a phase difference between the stress and strain due to hysteresis or internal friction damping (energy losses).

Stress level decays with distance.

4.4 Changes in Material Properties under Impact Loading







5.0 Elasto-Plastic Analysis

Yield Criteria

Tresca:

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_y}{2}$$
$$\frac{\sigma_2 - \sigma_3}{2} = \frac{\sigma_y}{2} \tag{5.1}$$
$$\frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y}{2}$$

Von Mises

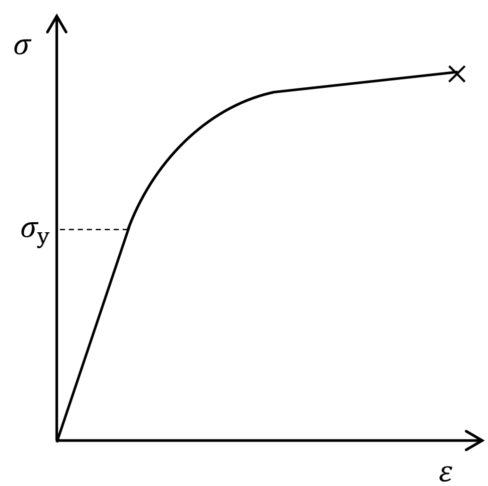
$$\frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{\frac{1}{2}} = \sigma_y \tag{5.2a}$$

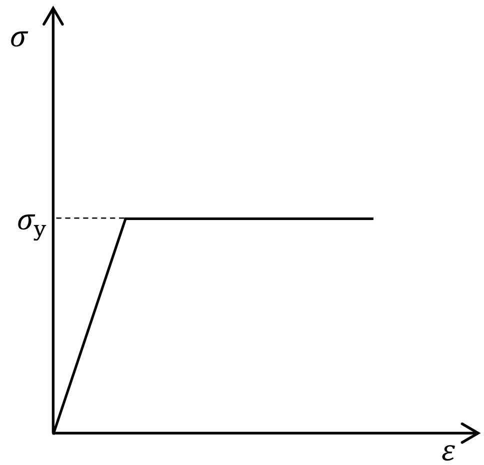
In 2D

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)^{\frac{1}{2}} = \sigma_y \tag{5.2b}$$

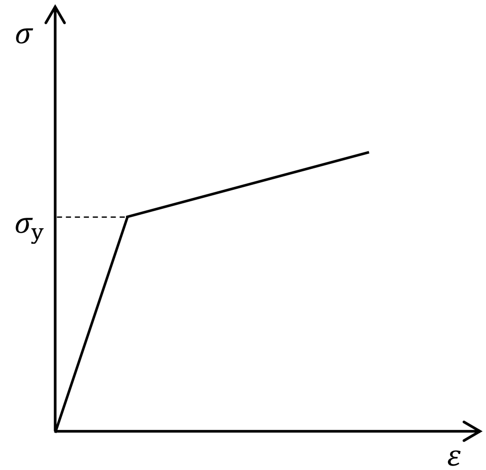
When the equivalent stress in parts of a component exceeds the yield stress of the material, elasto-plastic deformation occurs. Elasto-plastic analysis results in an analytical solution and is only possible for relatively few types of problems with simple geometry and idealized stress-strain behaviour (i.e. bending, torsion, axis-symmetric problems.)

General Stress-Strain Curve:

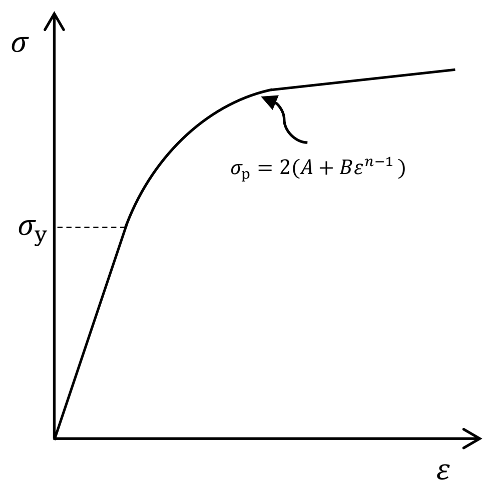




(a) elastic-perfectly plastic

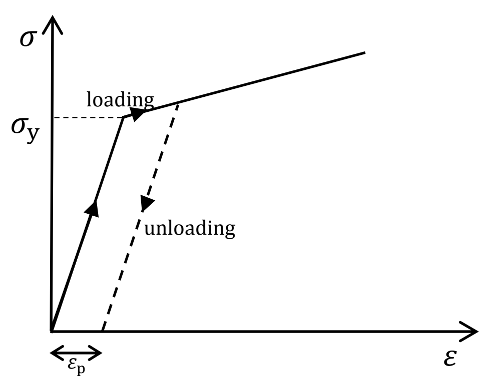


(b) elastic-linear plastic/hardening



***NOT COVERED IN THIS COURSE

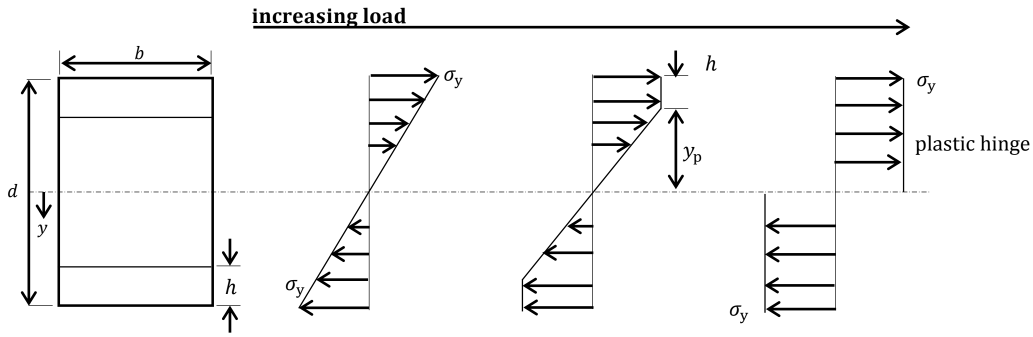
(c) non-linear



Permanent set occurs when the component is loaded beyond its yield point. When the load is removed there are residual stress. These can be both **beneficial** and **harmful**.

5.1 Beam Bending

Consider (a) and (b):



(i) onset of yielding (ii) elasto-plastic deformation (iii) fully plastic

$$\sigma = \frac{My}{I} \Rightarrow M = \frac{\sigma I}{y}$$

Elasto-plastic bending moment:

$$M_{\mathrm{ep}} = M_{\mathrm{e}} + M_{\mathrm{p}} \Rightarrow \underbrace{\frac{\sigma_{\mathrm{y}} b(d-2h)^2}{6}}_{\text{elastic}} + \underbrace{\sigma_{\mathrm{y}} bh(d-h)}_{\text{plastic}}$$

$$M_{\mathrm{ep}} = \frac{\sigma_{\mathrm{y}} bd^2}{6}\left[1 + 2\frac{h}{d}\left(1 - \frac{h}{d}\right)\right] \tag{5.3a}$$

$$M_{\mathrm{ep}} = \frac{\sigma_{\mathrm{y}} b}{12}\left[3d^2 - 4y_{\mathrm{p}}^2\right] \tag{5.3b}$$

distance from neutral axis

Fully plastic:

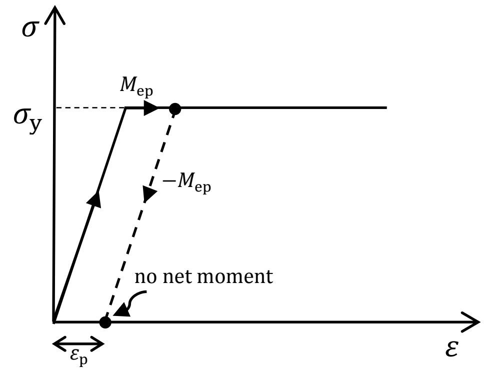
$$M_{\mathrm{ep}} = \frac{\sigma_{\mathrm{y}} bd^2}{6}\left[1 + 2\frac{1}{2}\left(1 - \frac{1}{2}\right)\right] \Rightarrow \sigma_{\mathrm{y}} = \frac{bd^2}{4}$$

$$M = \frac{\sigma I}{y} \Rightarrow \sigma_{\mathrm{y}} = \frac{bd^2}{6}$$

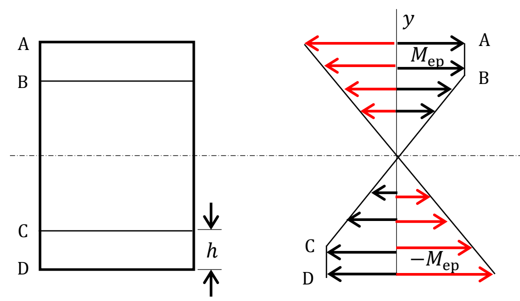
$$z_{\mathrm{p}}(\text{shape factor}) = \frac{M_{\mathrm{p}}}{M_{\mathrm{y}}} \Rightarrow 1.5$$

The shape factor five a measure of the increase in strength-carrying capacity available beyond the elastic limit.

Removal of loads after elasto-plastic deformation:

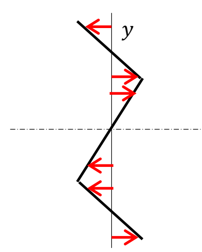


- equivalent to superposition of a linear-elastic unloading stress distribution over the initial stress distribution
- unloading stress distribution must be done to a moment of equal magnitude but opposite in direction of the applied load

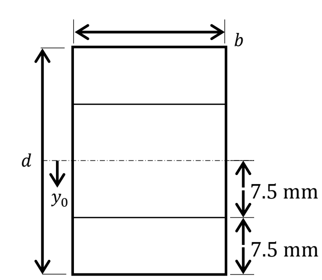


***not to scale

Subtract unloading from loading stress distribution to find residual stress distribution:



Example: Rectangular section beam loaded beyond elastic limit; yielding of half the cross section. Find the residual stress distribution when the load is removed.



Elastic-perfectly plastic; $\sigma_y = 600$ MPa; $b = 10$ mm; $d = 30$ mm

$$M_{ep} = \frac{\sigma_y b d^2}{6}\left[1 + 2\frac{h}{d}\left(1 - \frac{h}{d}\right)\right] \Rightarrow \frac{(600\times10^6)(10\times10^{-3})(30\times10^{-3})^2}{6}\left[1 + 2\frac{1}{4}\left(1 - \frac{1}{4}\right)\right]$$

$$\Rightarrow \boldsymbol{M_{ep} = 1\ 237.5\ \mathrm{N\cdot m}}$$

Elastic stresses due to M_{ep}:

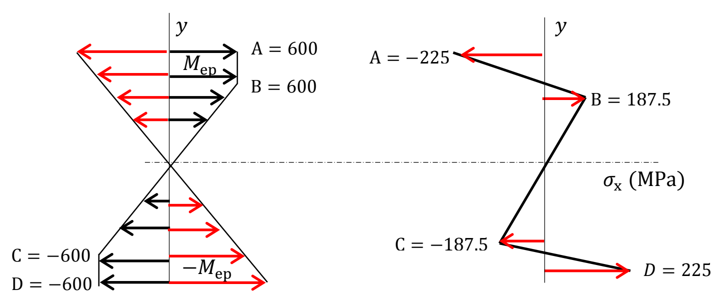
$$y = \frac{d}{2} \Rightarrow \sigma = \frac{-M_{ep}\frac{d}{2}}{I} \Rightarrow \boldsymbol{\sigma = -825.0\ \mathrm{MPa}}$$

$$y = y_0$$

$$\sigma = \frac{-M_{ep}\frac{d}{4}}{I} \Rightarrow \boldsymbol{\sigma = -412.5\ \mathrm{MPa}}$$

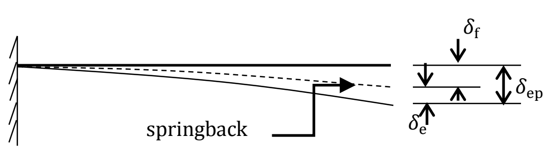
Residual stress distribution:

$$\begin{aligned}
&\text{A: } \sigma_x = -825.0 + 600 \Rightarrow -225\ \text{MPa} \\
&\text{B: } \sigma_x = -412.5 + 600 \Rightarrow 187.5\ \text{MPa} \\
&\text{C: } \sigma_x = 412.5 - 600 \Rightarrow -187.5\ \text{MPa} \\
&\text{D: } \sigma_x = 825. - 600 \Rightarrow 225\ \text{MPa}
\end{aligned}$$



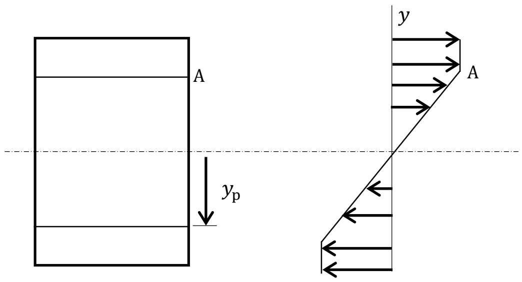
Springback refers to the elastic recovery of the plastically deformed structure when the loads are removed.

e.g.



$$\delta_f = \delta_{ep} - \delta_e$$

- strain: $\varepsilon = \frac{y}{R}$; R (radius of curvature)



$$\left.\varepsilon = \frac{y_{\mathrm{p}}}{R_{\mathrm{ep}}} \Rightarrow \frac{\sigma_{\mathrm{y}}}{\mathrm{E}}\right\} \text{ still elastic up to } y_{\mathrm{p}}$$

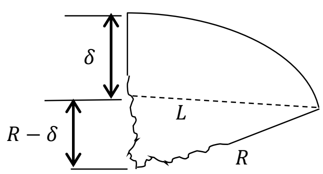
$$R_{\mathrm{ep}} = \frac{y_{\mathrm{p}}\mathrm{E}}{\sigma_{\mathrm{y}}} \Rightarrow y_{\mathrm{p}} = \frac{R_{\mathrm{ep}}\sigma_{\mathrm{y}}}{\mathrm{E}}$$

Substituting y_{p}back into equation (5.3b):

$$M_{\mathrm{ep}} = \frac{\sigma_{\mathrm{y}} b}{12}\left[3d^2 - 4\left(\frac{R_{\mathrm{ep}}\sigma_{\mathrm{y}}}{\mathrm{E}}\right)^2\right] \tag{5.4}$$

For elastic bending:

$$M_{\mathrm{e}} = \frac{\mathrm{EI}}{R_{\mathrm{e}}} \Rightarrow \frac{bd^3}{12}\frac{\mathrm{E}}{R_{\mathrm{e}}} \tag{5.5}$$



$$R^2 = L^2 + (R - \delta)^2$$

Since δ is small compared to R and L, neglect the δ^2 term.

$$L^2 \approx 2R\delta \Rightarrow \delta = \frac{L^2}{2R}$$

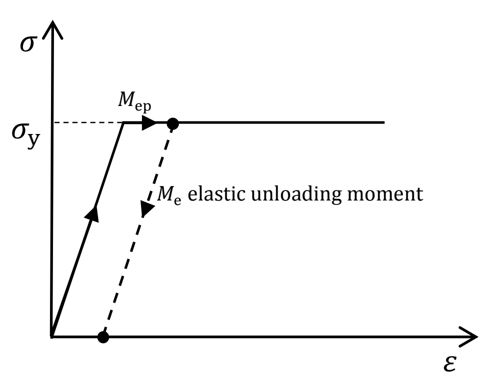
$$\delta_{\mathrm{f}} = \delta_{\mathrm{ep}} - \delta_{\mathrm{e}}$$

$$\frac{L^2}{2R_{\mathrm{f}}} = \frac{L^2}{2R_{\mathrm{ep}}} - \frac{L^2}{2R_{\mathrm{e}}};\ \ \frac{1}{R}\ \text{(curvature)}$$

$$\left.\frac{1}{R_{\mathrm{f}}} = \frac{1}{R_{\mathrm{ep}}} - \frac{1}{R_{\mathrm{e}}} \Rightarrow \frac{R_{\mathrm{ep}}}{R_{\mathrm{f}}} = 1 - \frac{R_{\mathrm{ep}}}{R_{\mathrm{e}}}\right\} \text{measure of springback; when } \frac{R_{\mathrm{ep}}}{R_{\mathrm{e}}} = 0 \Rightarrow \text{complete springback}$$

Since elastic loading after elastic-plastic deformation:

$$|M_{\mathrm{e}}| = |M_{\mathrm{ep}}|$$



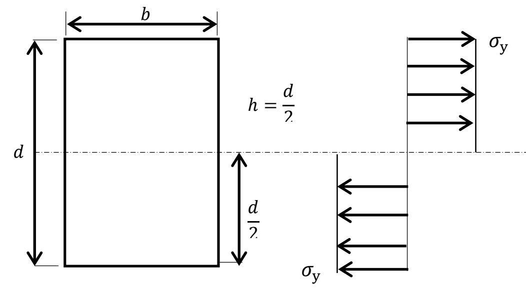
$$\frac{bd^3\mathrm{E}}{12R_\mathrm{e}} = \frac{\sigma_\mathrm{y} b}{12}\left[3d^2 - 4\left(\frac{R_\mathrm{ep}\sigma_\mathrm{y}}{\mathrm{E}}\right)^2\right]$$

$$\frac{R_\mathrm{ep}}{R_\mathrm{f}} = 1 - 3\left[\frac{R_\mathrm{ep}\sigma_\mathrm{y}}{\mathrm{E}d}\right] + 4\left[\frac{R_\mathrm{ep}\sigma_\mathrm{y}}{\mathrm{E}d}\right]^3 \quad (5.7)$$

$$\frac{R_\mathrm{ep}}{R_\mathrm{f}} = \left(\frac{R_\mathrm{ep}\sigma_\mathrm{y}}{\mathrm{E}d} + 1\right)\left(\frac{2R_\mathrm{ep}\sigma_\mathrm{y}}{\mathrm{E}d}\right)^2$$

In the fully plastic case:

$$M_\mathrm{p} = \frac{bd^2}{4}\sigma_\mathrm{y}$$



Elastic recovery:

$$|M_\mathrm{e}| = |M_\mathrm{ep}|$$

$$\frac{\mathrm{EI}}{R_\mathrm{e}} = \frac{bd^2}{4}\sigma_\mathrm{y}$$

$$\frac{R_\mathrm{p}}{R_\mathrm{f}} = \frac{1 - bd^2\sigma_\mathrm{y}R_\mathrm{p}}{4\mathrm{EI}}$$

Using $I = \frac{bd^3}{12}$

$$\frac{R_\mathrm{p}}{R_\mathrm{f}} = 1 - \frac{3\sigma_\mathrm{y}R_\mathrm{p}}{\mathrm{E}d} \quad (5.8)$$

Example: Determine the tool radius required to produce a final bend radius of 50 mm in a steel strip 20 mm wide and 2 mm thick. What is the required moment? Assume fully plastic deformation and against the assumptions.

$$\sigma_\mathrm{y} = 300\ \mathrm{MPa}; \quad \mathrm{E} = 200\ \mathrm{GPa}$$

$$\frac{R_\mathrm{p}}{R_\mathrm{f}} = 1 - \frac{3\sigma_\mathrm{y}R_\mathrm{p}}{\mathrm{E}d}$$

$$R_\mathrm{p} = (50 \times 10^{-3}\ \mathrm{m})\left[1 - \frac{3(300 \times 10^6\ \mathrm{Pa})R_\mathrm{p}}{(200 \times 10^9\ \mathrm{Pa})(2 \times 10^{-3}\ \mathrm{m})}\right] \Rightarrow \boldsymbol{R_\mathrm{p} = 44.9\ \mathrm{mm}}$$

$$M_\mathrm{p} = \frac{bd^2}{4}\sigma_\mathrm{y} \Rightarrow \frac{(20\times10^{-3}\ \mathrm{m})(2\times10^{-3}\ \mathrm{m})^2}{4}(300\times10^6\ \mathrm{Pa}) \Rightarrow \boldsymbol{M_\mathrm{p} = 6\ \mathrm{N\cdot m}}$$

Check the elastic case:

$$\frac{y_\mathrm{p}}{R_\mathrm{ep}} = \frac{\sigma_\mathrm{y}}{\mathrm{E}} \Rightarrow y_\mathrm{p} = \frac{\sigma_\mathrm{y} R_\mathrm{ep}}{\mathrm{E}} \Rightarrow \frac{(300\times10^6\ \mathrm{Pa})(44.9\times10^{-3}\ \mathrm{m})}{200\times10^9\ \mathrm{Pa}} \Rightarrow \boldsymbol{y_\mathrm{p} = 0.067\ \mathrm{mm}}$$

Since y_p is small, the assumption is correct.

Deflection

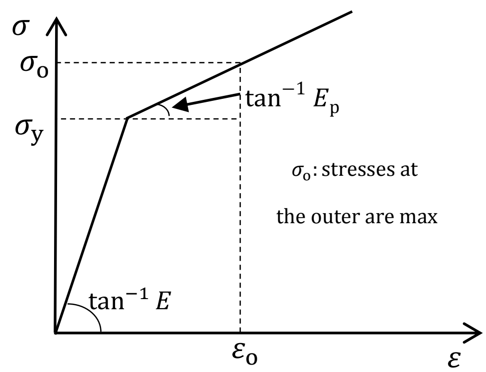
To find the deflection, use moment-curvature relationship as in the elastic case.

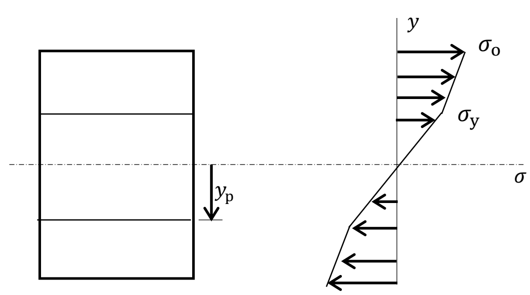
$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \Rightarrow M = -EI\frac{\partial^2 v}{\partial x^2}$$

The relationship between R_ep and M_ep is more complicated, integration has to be separated between the region that is plastically deformed and the region that is still elastic.

5.1.2 Elastic-Linear Strain Hardening

In





In the elastic range:

$$|y| < |y_\mathrm{p}|$$

$$\sigma_\mathrm{e} = \sigma_\mathrm{y}\frac{y}{y_\mathrm{p}} \tag{5.9a}$$

In the plastic region:

$$|y| > |y_\mathrm{p}|$$

$$\sigma_{\mathrm{p}} = \sigma_{\mathrm{y}} + (\sigma_{\mathrm{o}} - \sigma_{\mathrm{y}})\left(\frac{y - y_{\mathrm{p}}}{\frac{y}{2} - y_{\mathrm{p}}}\right) \tag{5.9b}$$

Using the stress-strain plot:

$$(\sigma_{\mathrm{o}} - \sigma_{\mathrm{y}}) = \left(\varepsilon_{\mathrm{o}} - \frac{\sigma_{\mathrm{y}}}{\mathrm{E}}\right)\mathrm{E_p}$$

$$\varepsilon_{\mathrm{o}} = \frac{\left(\frac{d}{2}\right)}{R_{\mathrm{ep}}}$$

$$\frac{y_{\mathrm{p}}}{R_{\mathrm{ep}}} = \frac{\sigma_{\mathrm{y}}}{\mathrm{E}}; \quad \text{when } y = y_{\mathrm{p}}$$

$$\varepsilon_{\mathrm{o}} = \frac{\left(\frac{d}{2}\right)\sigma_{\mathrm{y}}}{\mathrm{E}y_{\mathrm{p}}}$$

$$(\sigma_{\mathrm{o}} - \sigma_{\mathrm{y}}) = \frac{\mathrm{E_p}}{\mathrm{E}}\sigma_{\mathrm{y}}\left[\frac{\left(\frac{d}{2}\right)}{y_{\mathrm{p}}} - 1\right] \qquad \text{(5.10a) for } |y| > |y_{\mathrm{p}}|$$

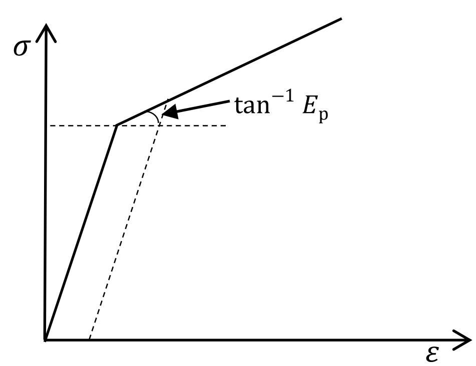
$$\sigma_{\mathrm{p}} = \sigma_{\mathrm{y}} + \frac{\mathrm{E_p}}{\mathrm{E}}\sigma_{\mathrm{y}}\left[\frac{y - y_{\mathrm{p}}}{y_{\mathrm{p}}}\right] \tag{5.10b}$$

Elastic-Plastic Bending Moment

$$M_{\mathrm{ep}} = \int_0^{\frac{d}{2}} \sigma\, by\, \mathrm{d}y \Rightarrow 2\left[\int_0^{y_{\mathrm{p}}} \sigma_{\mathrm{e}}\, by\, \mathrm{d}y + \int_{y_{\mathrm{p}}}^{\frac{d}{2}} \sigma_{\mathrm{p}}\, by\, \mathrm{d}y\right]$$

$$M_{\mathrm{ep}} = \frac{\sigma_{\mathrm{y}} b d^2}{12}\left[(3 - c^2) + \frac{\mathrm{E_p}}{c\mathrm{E}}(2 - 3c + c^3)\right]; \quad \text{where } c = \frac{y_{\mathrm{p}}}{d/2}$$

when $\frac{\mathrm{E_p}}{\mathrm{E}} = 0 \Rightarrow$ elastic-perfectly plastic case (5.3b)



For residual stress calculations, similar to the elastic-perfectly plastic case, assume elastic unloading due to applied $-M_{\mathrm{ep}}$.

- Superimpose elastic stress distribution due to $-M_{\mathrm{ep}}$ to that due to loading by M_{ep}.

Springback occurs when unloading from elasto-plastic deformation due to M_{ep}:

$$|M_{\mathrm{e}}| = |M_{\mathrm{ep}}|$$

$$\frac{1}{R_{\mathrm{e}}} = \frac{12M_{\mathrm{ep}}}{bd^3\mathrm{E}}$$

Moment to initiate yielding:

$$M_{\mathrm{y}} = \frac{bd^2}{6}\sigma_{\mathrm{y}}$$

$$\frac{1}{R_{\mathrm{e}}} = \frac{2\sigma_{\mathrm{y}}}{\mathrm{E}d}\frac{M_{\mathrm{ep}}}{M_{\mathrm{y}}}$$

$$\frac{1}{R_{\mathrm{e}}} = \frac{\sigma_{\mathrm{y}}}{\mathrm{E}}\left(\frac{1}{y_{\mathrm{p}}}\right)$$

$$\frac{1}{R_{\mathrm{f}}} = \frac{1}{R_{\mathrm{ep}}} - \frac{1}{R_{\mathrm{e}}}$$

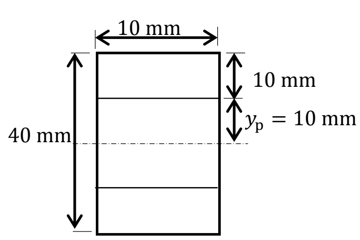
$$\frac{1}{R_{\mathrm{f}}} = \frac{\sigma_{\mathrm{y}}}{\mathrm{E}y} - \frac{2\sigma_{\mathrm{y}}}{\mathrm{E}d}\left(\frac{M_{\mathrm{ep}}}{M_{\mathrm{y}}}\right)$$

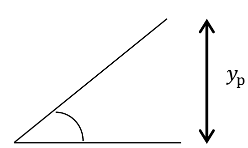
$$\frac{1}{R_{\mathrm{f}}} = \frac{2\sigma_{\mathrm{y}}}{\mathrm{E}d}\left(\frac{1}{c} - \frac{M_{\mathrm{p}}}{M_{\mathrm{y}}}\right); \quad \text{where } c = \frac{y_{\mathrm{p}}}{d/2} \qquad (5.13)$$

Example: A beam of rectangular cross-section ($b = 25$ mm, $d = 40$ mm) is loaded gradually so that the plastic zone spreads inwards from the outer fibers. Fine the values of the bending moment that cause:

a) yielding initiation
b) yielding up to a depth of 10 mm from the outer fibers
c) What is the final radius of curvature of the beam upon unloading?

$$\sigma_{\mathrm{y}} = 250 \text{ MPa}$$
$$\mathrm{E} = 200 \text{ GPa}$$
$$\mathrm{E_p} = 10 \text{ GPa (linear strain} - \text{hardening material)}$$





a)

$$M_y = \frac{bd^2}{6}\sigma_y \Rightarrow \frac{(25 \times 10^{-3}\ \text{m})(40 \times 10^{-3}\ \text{m})^2}{6}(250 \times 10^6\ \text{Pa}) \Rightarrow \boldsymbol{M_y = 1.667\ \text{kN} \cdot \text{m}}$$

b)

$$M_{ep} = \frac{\sigma_y bd^2}{12}\left[\left(3 - c^2\right) + \frac{E_p}{cE}\left(2 - 3c + c^3\right)\right]$$

$$c = \frac{y_p}{d/2} \Rightarrow \frac{10}{20} = 0.5$$

$$M_{ep} = \frac{(250 \times 10^6\ \text{Pa})\ (25 \times 10^{-3}\ \text{m})(40 \times 10^{-3}\ \text{m})^2}{12}\left[(3 - 0.5^2) + \frac{10}{0.5(250)}(2 - 3(0.5) + 0.5^3)\right]$$

$$\boldsymbol{M_{ep} = 2.8125\ \text{kN} \cdot \text{m}}$$

c)

$$\frac{1}{R_f} = \frac{2\sigma_y}{Ed}\left(\frac{1}{c} - \frac{M_p}{M_y}\right) \Rightarrow \frac{2(250 \times 10^6\ \text{Pa})}{(200 \times 10^9\ \text{Pa})(40 \times 10^{-3}\ \text{m})}\left[\frac{1}{0.5} - \frac{2.8125}{1.667}\right]$$

$$\frac{1}{R_f} = 0.0195\ \text{m}^{-1}$$

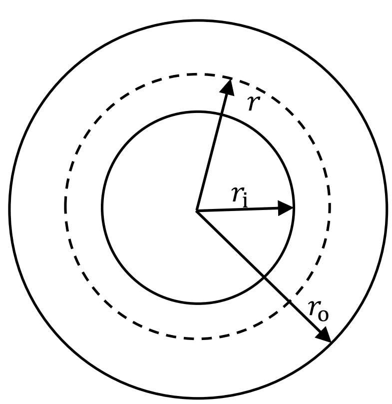
$$\boldsymbol{R_f = 51.2\ \text{m}}$$

Perfectly plastic: $\frac{E_p}{E} = 0$

$$\boldsymbol{M_{ep} = 2.2\ \text{kN} \cdot \text{m}}$$

$$\boldsymbol{R_f = 25.6\ \text{m}}$$

5.2 Thick-Walled Cylinders under Internal Pressure



Assume:

i) elastic-perfectly plastic

ii) Tresca's maximum shear stress yield criterion

$$\sigma_r = \frac{p_i}{k^2 - 1}\left(1 - \frac{r_o^2}{r^2}\right);\ \sigma_\theta = \frac{p_i}{k^2 - 1}\left(1 + \frac{r_o^2}{r^2}\right); k = \frac{r_o}{r_i}$$

In a thick-walled cylinder under internal pressure:

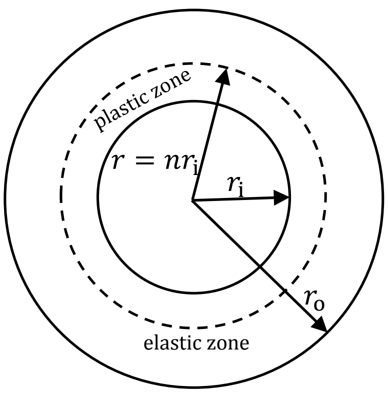
$$\sigma_\theta > \sigma_z > \sigma_r$$

$$\hat{\tau} = \frac{\sigma_\theta - \sigma_r}{2} \Rightarrow \frac{\sigma_y}{2} \Rightarrow \frac{p_i}{k^2 - 1}\left(\frac{r_o^2}{r^2}\right) \tag{5.15}$$

Yielding will occur first at the bore, where $r = r_i$

$$\hat{\tau} = \frac{p_i - k^2}{k^2 - 1} \Rightarrow \frac{\sigma_y}{2}$$

$$(p)_{\substack{\text{elastic}\\\text{breakdown}}} = \frac{k^2 - 1}{2k^2}\sigma_y \tag{5.16}$$



- consider partial yielding in the cylinder to a depth equivalent to a radius ratio if n (i.e. yielding at $r = r_i$ to $r = nr_i$)

Let the radial pressure at $r = nr_i$ to be equal to P_n

- consider the elastic zone

substitute for k the quantity $\frac{r_o}{nr_i} = \frac{k}{n}$

$$\hat{\tau} = \frac{P_n}{\left(\frac{k}{n}\right)^2 - 1}\left(\frac{k}{n}\right)^2 \Rightarrow \frac{\sigma_y}{2}$$

$$P_n = \frac{\sigma_y\left(k^2 - n^2\right)}{2k^2} \tag{5.17}$$

Consider the plastic zone equilibrium equation:

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0$$

In the plastic zone:

$$\hat{\tau} = \frac{\sigma_\theta - \sigma_r}{2} \Rightarrow \frac{\sigma_y}{2}$$

$$r\frac{d\sigma_r}{dr} + \sigma_y = 0 \Rightarrow d\sigma_r = \sigma_y \frac{dr}{r}$$

Integrating

$$\sigma_r = \sigma_y \ln r + C$$

Boundary condition at $r = nr_i \Rightarrow \sigma_r = -p_n$

$$C = -p_n - \sigma_y \ln(nr_i) \Rightarrow C = \frac{-\sigma_y\left(k^2 - n^2\right)}{2k^2} - \sigma_y \ln(nr_i)$$

$$\sigma_r = \sigma_y\left[\ln\left(\frac{r}{nr_i}\right) - \frac{(k^2 - n^2)}{2k^2}\right] \tag{5.18a}$$

$$\frac{\sigma_\theta - \sigma_r}{2} = \frac{\sigma_y}{2} \Rightarrow \sigma_\theta = \sigma_y + \sigma_r$$

$$\sigma_\theta = \sigma_y\left[\ln\left(\frac{r}{nr_i}\right) - \frac{(k^2 + n^2)}{2k^2}\right] \tag{5.18b}$$

when $r = r_i \Rightarrow \sigma_r = -p$

From (5.18a)

$$p = \sigma_y\left[\ln(n) + \frac{(k^2 - n^2)}{2k^2}\right] \tag{5.19}$$

p is the pressure required to cause yielding to the radial depth nr_i so when $n = k$, complete yielding throughout the cylinder is present.

$$p = p_c \Rightarrow \sigma_y \ln k \tag{5.20}$$

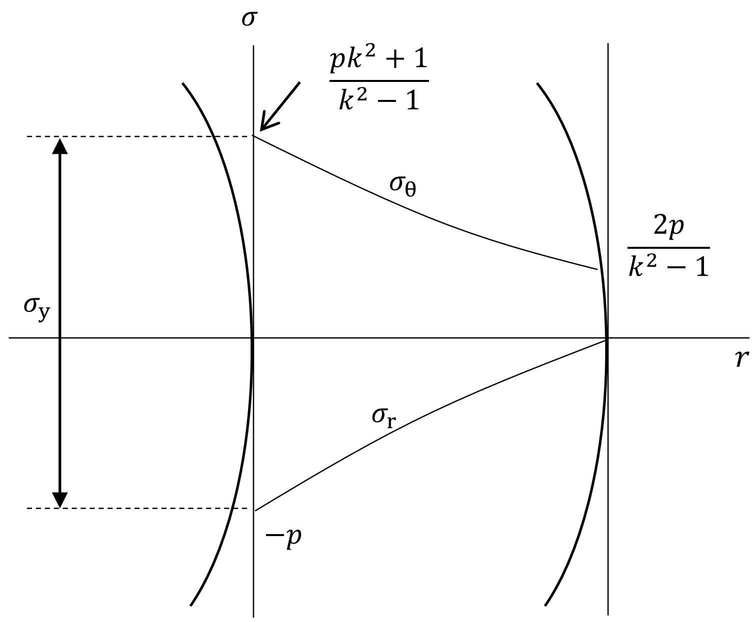
p_c: collapse pressure

At the collapse pressure:

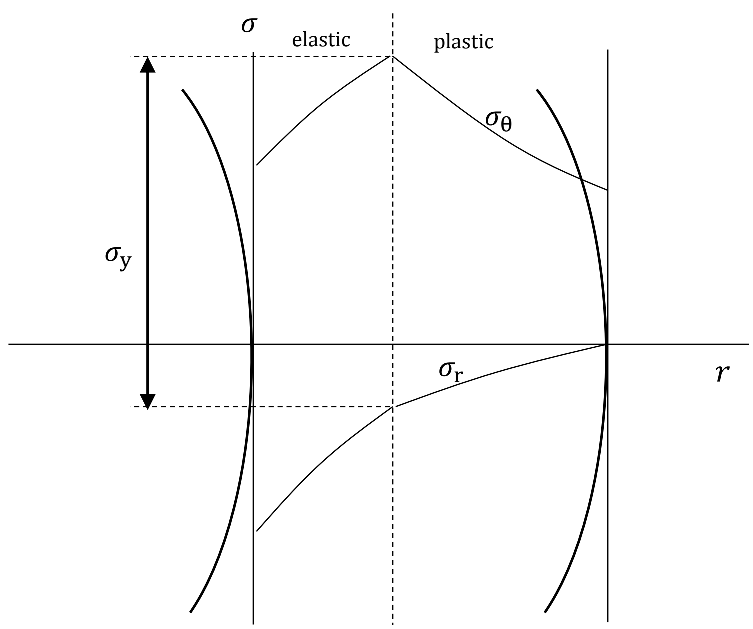
$$(\sigma_r)_{p_c} = \sigma_y \ln\left(\frac{r}{r_o}\right)$$
$$(\sigma_\theta)_{p_c} = \sigma_y\left[1 + \ln\left(\frac{r}{r_o}\right)\right] \tag{5.21}$$

The stress distribution:

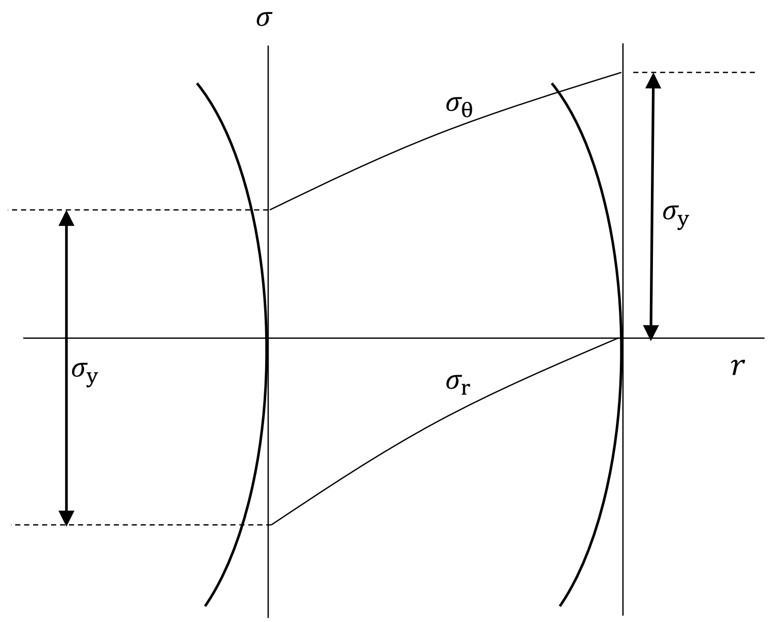
a) onset of yielding at the bore



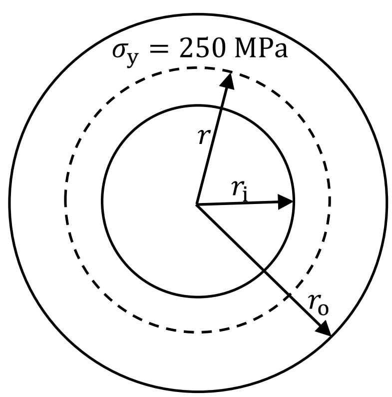
b) elastic-plastic cylinder



c) fully plastic cylinder



Example: elastic-fully plastic, $k = 2, p = -\sigma_r$



a) elastic breakdown pressure

b) the pressure to cause yielding up to 50% across its thickness, and the corresponding stresses σ_r and σ_θ at the elastic-plastic interface

c) the collapse pressure and the corresponding hoop stresses at the bore

a) p_{eb}

$$p_{eb} = \frac{k^2 - 1}{2k^2}\sigma_y \Rightarrow \frac{4-1}{2(4)}(250 \text{ MPa}) \Rightarrow \boldsymbol{p_{eb} = 93.75 \text{ MPa}}$$

b) p

$$p = \sigma_y\left[\ln(n) + \frac{(k^2 - n^2)}{2k^2}\right]; n = 1.5$$

$$p = (250 \text{ MPa})\left[\ln(1.5) + \frac{(2^2 - 1.5^2)}{2(2^2)}\right] \Rightarrow \boldsymbol{p = 156.1 \text{ MPa}}$$

At the interface:

$$p_{\mathrm{n}} = \sigma_{\mathrm{y}} \left[\frac{(k^2 - n^2)}{2k^2}\right]; n = 1.5$$

$$p_{\mathrm{n}} = (250\ \mathrm{MPa}) \left[\frac{(2^2 - 1.5^2)}{2(2^2)}\right] \Rightarrow \boldsymbol{p_{\mathrm{n}} = 54.7\ \mathrm{MPa}}$$

$$(\sigma_{\mathrm{r}})_{n=1.5} = -p_{\mathrm{n}} \Rightarrow -\mathbf{54.7\ MPa}$$

$$(\sigma_{\theta})_{n=1.5} = \sigma_{\mathrm{y}} + \sigma_{\mathrm{r}} \Rightarrow 250 - 54.7 \Rightarrow \mathbf{195.3\ MPa}$$

c) p_{c}

$$p_{\mathrm{c}} = \sigma_{\mathrm{y}} \ln k \Rightarrow (250\ \mathrm{MPa}) \ln 2 \Rightarrow \mathbf{173.2\ MPa}$$

$$\sigma_{\theta} = \sigma_{\mathrm{y}} + \sigma_{\mathrm{r}} \Rightarrow 250 - 173.2 \Rightarrow \mathbf{76.7\ MPa}$$

5.2.1 Residual Stresses in Internally Pressurized Cylinders

Compounding cylinders is one way to obtain favourable pre-stresses before they are put into service. This is a more effective use of material. An even more effective use of material, while achieving the same goal, is by applying enough pressure to cause yielding (residual stresses) in some or all of the material in a simple cylinder then releasing the pressure before it is put into service. The generation of favourable residual stresses in a cylinder by plastic action is called **autofrettage.**

Consider a fully (100% overstrained) cylinder which is being unloaded after reaching the state.

Assume elastic unloading.

Stresses due to the applied collapse pressure, P_{c}:

$$\begin{aligned} (\sigma_{\mathrm{r}})_{p_{\mathrm{c}}} &= \sigma_{\mathrm{y}} \ln\left(\frac{r}{r_{\mathrm{o}}}\right) \\ (\sigma_{\theta})_{p_{\mathrm{c}}} &= \sigma_{\mathrm{y}} \left[1 + \ln\left(\frac{r}{r_{\mathrm{o}}}\right)\right] \end{aligned} \qquad (5.21)$$

For elastic unloading due to the P_{c}

$$\sigma_{\mathrm{r}} = \frac{p_{\mathrm{c}}}{k^2 - 1}\left(1 - \frac{r_{\mathrm{o}}^2}{r^2}\right)$$

$$\sigma_{\theta} = \frac{c}{k^2 - 1}\left(1 + \frac{r_{\mathrm{o}}^2}{r^2}\right)$$

$$(\sigma_{\mathrm{r}})_{\mathrm{res}} = \sigma_{\mathrm{y}} \ln\left(\frac{r}{r_{\mathrm{o}}}\right) - \frac{p_{\mathrm{c}}}{k^2 - 1}\left(1 - \frac{r_{\mathrm{o}}^2}{r^2}\right)$$

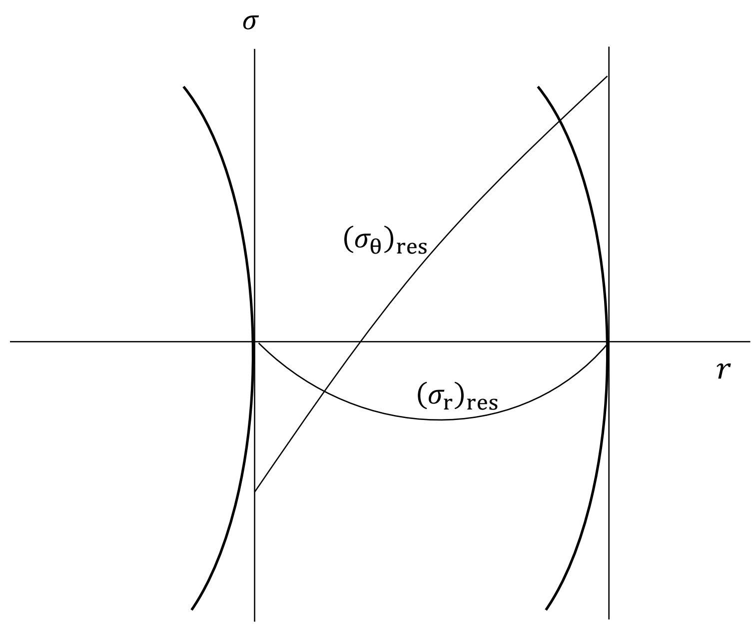
$$(\sigma_{\theta})_{\mathrm{res}} = \sigma_{\mathrm{y}} \left[1 + \ln\left(\frac{r}{r_{\mathrm{o}}}\right)\right] - \frac{p_{\mathrm{c}}}{k^2 - 1}\left(1 + \frac{r_{\mathrm{o}}^2}{r^2}\right)$$

$p_c = \sigma_y \ln(k)$

$$(\sigma_r)_{res} = \sigma_y \left[\ln\left(\frac{r}{r_o}\right) - \frac{\ln k}{k^2 - 1}\left(1 - \frac{r_o^2}{r^2}\right)\right] \tag{5.22a}$$

$$(\sigma_\theta)_{res} = \sigma_y \left[1 + \ln\left(\frac{r}{r_o}\right) - \frac{\ln k}{k^2 - 1}\left(1 + \frac{r_o^2}{r^2}\right)\right] \tag{5.22b}$$

$$\hat{\tau}_{res} = \frac{|(\sigma_\theta)_{res}| - |(\sigma_r)_{res}|}{2} \Rightarrow \frac{\sigma_y}{2}\left[1 - \frac{2\ln k}{k^2 - 1}\left(\frac{r_o^2}{r^2}\right)\right] \tag{5.23}$$



Reverse Yielding:

$$\hat{\tau}_{res} = -\hat{\tau}_y \Rightarrow -\frac{\sigma_y}{2}$$

$$\frac{\sigma_y}{2}\left[1 - \frac{2\ln k}{k^2 - 1}\left(\frac{r_o^2}{r^2}\right)\right] = -\frac{\sigma_y}{2}$$

Max stresses occur at the bore:

$$1 - \frac{2\ln k}{k^2 - 1}\ln k = -1 \Rightarrow \boldsymbol{k = 2.22}$$

$k = 2.22$ is the limiting radius above which 100% overstrain (fully plastic condition) will give rise to reverse yield at the bore

100% autofrettage: the theoretical max residual stress left in the bore

50% autofrettage: 50% theoretical residual stress left in the bore

5.2.2 The “Shakedown” Condition

- reverse yielding is not desirable
- “shakedown” $\Rightarrow$ cylinder settling down to a safe working condition after initial plastic deformation
- Optimum (100%) autofrettage is achieved with 100% overstrain when $k \leq 2.22$ but if $k > 2.22$ with less that 100% overstrain

Max allowable pressure ($\hat{P}_{\mathrm{i}}$) for $k \leq 2.22$

$$\frac{\hat{P}_{\mathrm{i}} k^2}{k^2-1} + \frac{\sigma_{\mathrm{y}}}{2}\left[1 - \frac{2\ln k}{k^2-1}k^2\right] = \frac{\sigma_{\mathrm{y}}}{2}$$

Due to $\hat{P}_{\mathrm{i}}$

$$\hat{P}_{\mathrm{i}} = \sigma_{\mathrm{y}} \ln k \Rightarrow P_{\mathrm{c}} \tag{5.24}$$

$$\frac{\hat{P}_{\mathrm{i}}}{P_{\mathrm{eb}}}$$

$$P_{\mathrm{eb}} = \frac{k^2-1}{2k^2}\sigma_{\mathrm{y}}; \text{for } k = 2.22 \Rightarrow \frac{\hat{P}_{\mathrm{i}}}{P_{\mathrm{eb}}} \leq 2$$

Load-carrying capacity increased by a factor of 2.

For $k > 2.22$

$$\frac{\hat{P}_{\mathrm{i}} k^2}{k^2-1} - \frac{\sigma_{\mathrm{y}}}{2} = \frac{\sigma_{\mathrm{y}}}{2}$$

$$\hat{P}_{\mathrm{i}} = \frac{k^2}{k^2-1}\sigma_{\mathrm{y}} \Rightarrow 2P_{\mathrm{eb}}$$

Load-carrying capacity is increased by a factor of 2 for any $k > 2.22$.

Example (a):

Determine – for completely linear-elastic behaviour of a cylinder with radius ratio k=2.5 and $\sigma_{\mathrm{y}} =$ 400 MPa – the optimum autofrettage pressure and the resulting percentage overstain of the cylinder wall.

$$\hat{P}_{\mathrm{i}} = 2P_{\mathrm{eb}} \Rightarrow 2\frac{k^2-1}{k^2}\frac{\sigma_{\mathrm{y}}}{2} \Rightarrow \frac{2.5^2-1}{2.5^2}400 = 336 \text{ MPa}$$

$$\text{overstrain:} \frac{n-1}{k-1} \times 100\%$$

$$\sigma_r = \sigma_y \left[\ln\left(\frac{r}{nr_i}\right) - \frac{(k^2 - n^2)}{2k^2} \right] \tag{5.18a}$$

$$@r = r_i, -\hat{P}_i = \sigma_r \Rightarrow -336 \text{ MPa}$$

$$-336 \text{ MPa} = 400 \left[\ln\left(\frac{1}{n}\right) - \frac{(2.5 - n^2)}{2(2.5)^2} \right]$$

$$\ln\left(\frac{1}{n}\right) - \frac{(2.5 - n^2)}{2(2.5)^2} = -\frac{336}{400} \Rightarrow \text{f}(n)$$

n	f(n)
1.5	0.725
2	0.875
1.75	0.815

$$\therefore n = 1.845$$

$$\%\text{ overstrain: } \frac{1.845 - 1}{2.5 - 1} \times 100\,\% \Rightarrow 56.3\,\%$$

Example (b):

A thick-walled steel cylinder pressure vessel, optimally autofrettaged for linear-elastic behaviour in service, has inner and outer diameters of 110 mm and 160 mm, respectively. What are the radial and hoop stresses at the inner and outer radii of the cylinder when it is subjected to an internal pressure of 200 MPa.

$$\sigma_y = 660 \text{ MPa; assume elastic-perfectly plastic}$$

$$k = \frac{r_o}{r_i} \Rightarrow \frac{55}{80} = 1.4545 \le 2.22$$

$$\hat{P}_i = P_c \Rightarrow \sigma_y \ln k = 600 \ln(1.4545) \Rightarrow 247.3 \text{ MPa}$$

$\hat{P}_i$: optimal autofrettage pressure

$$(\sigma_r)_{res} = \sigma_y \left[\ln\left(\frac{r}{r_o}\right) - \frac{\ln k}{k^2 - 1}\left(1 - \frac{r_o^2}{r^2}\right) \right]$$

$$(\sigma_\theta)_{res} = \sigma_y \left[1 + \ln\left(\frac{r}{r_o}\right) - \frac{\ln k}{k^2 - 1}\left(1 + \frac{r_o^2}{r^2}\right) \right]$$

@ r_i and r_o

$$(\sigma_r)_{res} = 0$$

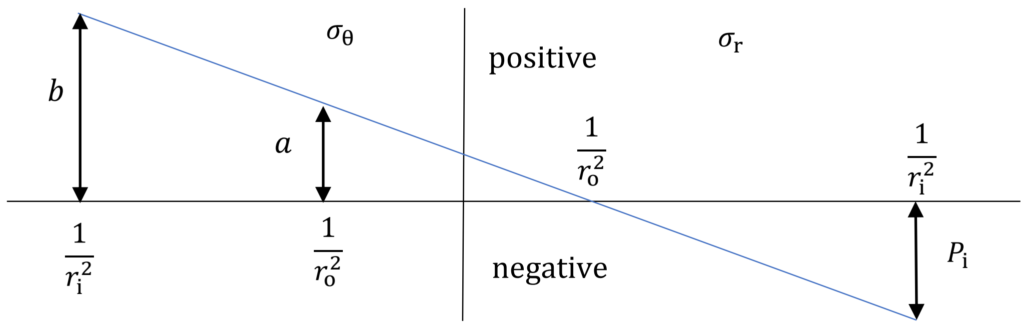
$$(\sigma_\theta)_{r_i} = \sigma_y \left[1 - \frac{(2k^2)}{k^2 - 1} \ln k \right] \Rightarrow -278 \text{ MPa}$$

$$(\sigma_\theta)_{r_o} = \sigma_y \left[1 + \ln\left(\frac{r}{r_o}\right) - \frac{\ln k}{k^2 - 1}\left(1 + \frac{r_o^2}{r^2}\right) \right] \Rightarrow 600(0.3283) = +216.7 \text{ MPa}$$

Elastic stresss due to $P_\mathrm{i} = 200$ MPa $\Rightarrow$ Lamé plot

$$\frac{1}{r_\mathrm{i}^2} = 330.5$$

$$\frac{1}{r_\mathrm{o}^2} = 156.25$$



$$\frac{b}{\left(\frac{1}{r_\mathrm{i}^2} + \frac{1}{r_\mathrm{o}^2}\right)} = \frac{a}{2\frac{1}{r_\mathrm{o}^2}} \Rightarrow \frac{P_\mathrm{i}}{\frac{1}{r_\mathrm{i}^2} - \frac{1}{r_\mathrm{o}^2}}$$

$$(\sigma_\theta)_{r_\mathrm{o}} = a \Rightarrow 858.5 \text{ MPa}$$
$$(\sigma_\theta)_{r_\mathrm{i}} = b \Rightarrow 558.5 \text{ MPa}$$

Total stress in the autofrettage cylinder $P_\mathrm{i} = 200$ MPa

$@\, r = r_\mathrm{i}$

$$\sigma_\mathrm{r} = -200 \text{ MPa}$$
$$\sigma_\theta = 558.5 - 278 \Rightarrow 280.5 \text{ MPa}$$

$@\, r = r_\mathrm{o}$

$$\sigma_\mathrm{r} = 0 \text{ MPa}$$
$$(\sigma_\theta)_{r_\mathrm{o}} = (\sigma_\theta)_{r_\mathrm{o}} + \left[(\sigma_\theta)_{r_\mathrm{o}}\right]_\mathrm{res} \Rightarrow 358.5 + 216.7$$
$$(\sigma_\theta)_{r_\mathrm{o}} = 575.2 \text{ MPa}$$

5.2.3 Using von Mises

- If von Mises criteria is used, $\tau_\mathrm{y} = \frac{\sigma_\mathrm{y}}{\sqrt{3}}$, instead of $\tau_\mathrm{y} = \frac{\sigma_\mathrm{y}}{2}$ in Tresca.
- All the expressions developed for Tresca can be used when σ_y is replaced by $\frac{2}{\sqrt{3}}\sigma_\mathrm{y}$.