MAT 1322

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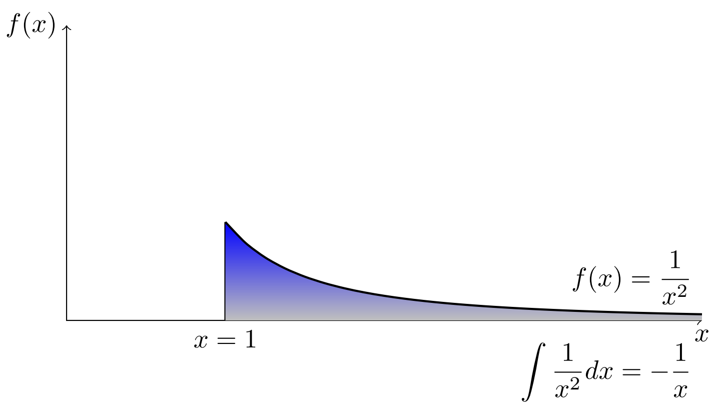
Chapter 5 : Integrals

5.10 : Improper Integrals

Consider $f(x) = \dfrac{1}{x^2}$ and let's try to find $\displaystyle\int_1^\infty \frac{1}{x^2}dx$.



This integral would represent the area under the curve from 1 to ∞. Is this finite or not? We'll look at this another way: we'll consider

$$\int_1^t \frac{1}{x^2}dx$$

and let t get bigger. We have that

So if:

$$\begin{array}{lll} t = 100 & \displaystyle\int_1^t \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^{100} & = -\dfrac{1}{100} + 1 = 0.99 \\ t = 10000 & \displaystyle\int_1^t \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^{10000} & = -\dfrac{1}{10000} + 1 = 0.9999 \\ t = 1000000 & \displaystyle\int_1^t \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^{1000000} & = -\dfrac{1}{1000000} + 1 = 0.999999 \end{array}$$

It looks like $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ should be equal to 1, but how do we verify this properly? By formalizing what we just did above.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} -1\frac{1}{x}\Big|_1^t = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1$$

What about if $f(x) = \dfrac{1}{\sqrt{x}}$?

$$\text{then } \int_1^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty} 2\sqrt{x}\Big|_1^t = \lim_{t\to\infty} 2\sqrt{t} - 2 \to \infty$$

But how does this happen? Doesn't the graph of $f(x) = \dfrac{1}{\sqrt{x}}$ look like the graph of $f(x) = \dfrac{1}{x^2}$? How can one of them have finite area and the other not?

REASON: $\dfrac{1}{x^2}$ *goes to zero much faster than* $f(x) = \dfrac{1}{\sqrt{x}}$.

Let's look at the "general" problem $\displaystyle\int_1^\infty \frac{1}{x^p}dx$.

If $p \neq 1$, then

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}dx = \lim_{t\to\infty} \frac{1}{-p+1}\left(t^{-p+1} - 1\right)\Big|_1^t = \lim_{t\to\infty} \frac{1}{-p+1}\left(t^{-p+1} - 1\right)$$

So if $p > 1$, then

$$\lim_{t\to\infty} t^{-p+1} = 0$$

and so

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$$

If $p < 1$, then

$$\lim_{t\to\infty} t^{-p+1} \to \infty$$

and so

$$\int_1^\infty \frac{1}{x^p}dx \to \infty$$

If $p = 1$, then

$$\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln t - \ln 1 \to \infty$$

An integral of the form $\displaystyle\int_a^b f(x)dx$ is said to be <u>improper</u> (of type I) if $a = -\infty$ and/or $b = \infty$.

$\displaystyle\int_a^\infty f(x)dx$ (and $\displaystyle\int_{-\infty}^b f(x)dx$) are said to be <u>convergent</u> if the limit $\displaystyle\lim_{t\to\infty} \int_t^b f(x)dx$ exist (i.e finite) and <u>divergent</u> if the limit doesn't exist.

So we've shown that $\displaystyle\int_1^\infty \frac{1}{x^p}dx$ converges to $\dfrac{1}{p-1}$ if $p > 1$ and diverges if $p \leq 1$. What do we do with $\displaystyle\int_{-\infty} \infty f(x)dx$? Use finite number a and split the integral the following way:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

The only way that $\int_{-\infty}^{\infty} f(x)dx$ can be finite is if <u>both</u> $\lim_{t\to-\infty}\int_t^a f(x)dx$ and $\lim_{t\to\infty}\int_a^t f(x)dx$ converge. If either diverges, then $\int_{-\infty}^{\infty} f(x)dx$ diverges.

Ex. 2 and 3 (p.416-7)

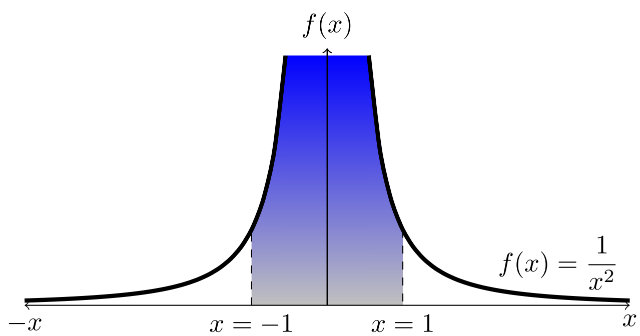
There is a second type of Improper Integral:

What's $\int_{-1}^{1}\frac{1}{x^2}dx$? Let's see

$$\int_{-1}^{1}\frac{1}{x^2}dx = -\frac{1}{x}\bigg|_{-1}^{1} = -1-1 = 2$$

Fine?

No, how can the area under a positive curve be negative? So there's something else going on here.



Remember $\frac{1}{x^2} \to \infty$ as $x \to 0$! This is the second type of Improper Integral (when $f(x)$ is unbounded on $[a, b]$).

So what do we do? We have to split the integral into 2 pieces and look at each separately, i.e:

$$\int_{-1}^{1}\frac{1}{x^2}dx = \int_{-1}^{0}\frac{1}{x^2}dx + \int_{0}^{1}\frac{1}{x^2}dx$$

By the symmetry of the function, $\frac{1}{x^2}$ is even, we only have to look at one of them since

$$\int_{-1}^{1}\frac{1}{x^2}dx = 2\int_{0}^{1}\frac{1}{x^2}dx \qquad \left(f(x)\text{ even, }\int_{-a}^{a}\frac{1}{x^2}dx = 2\int_{0}^{a}\frac{1}{x^2}dx\right)$$

But how do we do $\int_0^1\frac{1}{x^2}dx$? The same way that we did the previous integrals, we'll look at $\lim_{t\to0}\int_t^1\frac{1}{x^2}dx$ but now, we're approaching 0 from the right, so we denote this by

$$\lim_{t\to0^+}\int_t^1\frac{1}{x^2}dx = \lim_{t\to0^+} -\frac{1}{x}\bigg|_t^1 = \lim_{t\to0^+} -1+\frac{1}{t} = \infty$$

$\therefore$ we can say that $\int_{-1}^{1}\frac{1}{x^2}dx$ diverges.

What about $\int_0^1\frac{1}{\sqrt{x}}dx$?

$$\int_0^1\frac{1}{\sqrt{x}}dx = \lim_{t\to0^+}\int_t^1\frac{1}{\sqrt{x}}dx = \lim_{t\to0^+} 2\sqrt{x}\big|_t^1 = \lim_{t\to0^+} 2-2\sqrt{t} = 2$$

so $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges.

An integral of the form $\int_a^b f(x)dx$ where $f(x)$ has a discontinuity (typically a vertical assymptote) somewhere on the interval of integration is called an improper integral (of type II) . Example:

$$\int_0^4 \frac{1}{(x-2)^4} dx \qquad f(x) = (x-2)^4 \to \infty \text{ as } x \to 2$$

So we write

$$\int_0^4 \frac{1}{(x-2)^4} dx = \int_0^2 \frac{1}{(x-2)^4} dx + \int_2^4 \frac{1}{(x-2)^4} dx$$

Then:

$$\begin{aligned} \int_0^2 \frac{1}{(x-2)^4} dx &= \lim_{t \to 2^-} \int_0^t \frac{1}{(x-2)^4} dx \qquad \text{(approaching 2 from the left)} \\ &= \lim_{t \to 2^-} -\frac{1}{3}(x-2)^{-3}\Big|_0^t \\ &= \lim_{t \to 2^-} -\frac{1}{3}\left[(t-2)^{-3} - (-2)^{-3}\right] \\ &= \lim_{t \to 2^-} -\frac{1}{3}\left[\frac{1}{(t-2)^3} + \frac{1}{8}\right] \\ &= \infty \end{aligned}$$

$\therefore \int_0^4 \frac{1}{(x-2)^4} dx$ diverges. See also examples 5-8 on pages 418-20.

Sometimes, we would be unable to find the exact value of an improper integral (eg unable to find the antiderivative) but we would still like to know whether the integral is convergent or divergent. We can do this by making comparisons.

The Comparison Test for $\int_a^\infty f(x)dx$: (assuming $f(x) \geq 0$):

1. Guess wheter or not the integral converges or diverges by looking at the behavior of $f(x)$ for large x (ie do you think $f(x) \to 0$ fast enough to converges?).

2. Confirm the guess by comparison. If:

$$0 \leq g(x) \leq f(x)$$

   and:

$$\int_a^\infty f(x)dx$$

   converges, then :

$$\int_a^\infty g(x)dx$$

   converges

Comparison Theorem. Because function are positive, $\int_a^b g(x)dx \leq \int_a^b f(x)dx$, if:

$$0 \leq g(x) \leq f(x)$$

and:

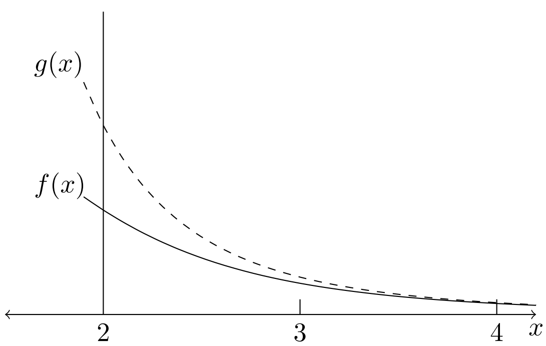
$$\int_a^\infty g(x)dx$$

diverges, then :

$$\int_a^\infty f(x)dx$$

diverges.

Example 1 : does $\int_2^\infty \frac{dx}{x^4+3x+7}$ converge or not ?



As $x \to \infty$, $x^4 + 3x + 7$ goes like x^4, so

$$\frac{1}{x^4+3x+7}$$

goes to zero rapidly, so we'd expect the integral to converge.

Now, we need to find the appropriate function for comparison. For $x \geq 2$, $x^4+3x+7 > x^4 > 0$ (i.e $3x+7 > 0$).

So we have:

$$\frac{1}{x^4+3x+7} < \frac{1}{x^4}$$

and since $\int_1^\infty \frac{1}{x^4}dx$ converges, $\int_2^\infty \frac{1}{x^4}dx$ must converge and so $\int_2^\infty \frac{1}{x^4+3x+7}dx$ converges by comparison.

Example 2 : what about $\int_1^\infty \frac{2-\cos x}{x}dx$? Always have $-1 \leq \cos x \leq 1$, let's see. **Ex 9-10 p.420-1**

Since $-1 \leq \cos x \leq 1$;

$$\frac{1}{x} \leq \frac{2-\cos x}{x} \leq \frac{3}{x}$$

so suspect the integral diverges, and we know that

$$\int_1^\infty \frac{dx}{x}$$

diverges, so our integral must diverge as well.

Chapter 6 : Applications of Integration

6.1 : More about areas

What is the area of the region the lies between $y = f(x)$ and $y = g(x)$ (where $f(x) \geq g(x)$ in $[a, b]$) between $x = a$ and $x = b$?

If $f(x) \geq 0$ and $g(x) \geq 0$ for x in $[a, b]$, then it's clear that this area is the area under $f(x)-$area under $g(x)$. This area is

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\ dx$$

But it really doesn't matter that $f(x) \geq 0$ and $g(x) \geq 0$, integral still works the same way regardless of the signs of f and g.

The area is $\int_a^b (f(x) - g(x))\ dx$ because we can move the curves up and down and it doesn't change anything.

Examples:

1. Find the area of the region bounded by the curves $y = \ln x, y = x$ and the lines $x = 2$ and $x = 3$.

$$\begin{aligned}
\text{area} &= \int_2^3 (x - \ln x)\ dx \\
&= \frac{1}{2}x^2\Big|_2^3 - \int_2^3 \ln x\,dx \\
u &= \ln x \qquad du = \frac{1}{x}\,dx \\
dv &= dx \qquad v = x \\
&= \left(\frac{1}{2}x^2 - (x\ln x - x)\right)\Big|_2^3 \\
&= \left(\frac{1}{2}(3)^2 - 3\ln 3 + 3\right) - \left(\frac{1}{2}(2)^2 - 2\ln 2 + 2\right) \\
&\approx 1.59
\end{aligned}$$

2. Find the area bounded by the curves $y = x^2$ and $y = 4 - x^2$.

The parabolas intersect when

$$\begin{aligned}4 - x^2 &= x^2\\ 4 &= 2x^2\\ x^2 &= 2\\ x &= \pm\sqrt{2}\end{aligned}$$

so the area is

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}} \left((4-x^2)-(x^2)\right)\,dx &= \int_{-\sqrt{2}}^{\sqrt{2}} \left(4-2x^2\right)\,dx\\ &= \left(4x - \frac{2}{3}x^3\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}\\ &= \left(4\sqrt{2} - \frac{2}{3}\left(2\sqrt{2}\right)\right) - \left(-4\sqrt{2} + \frac{2}{3}\left(2\sqrt{2}\right)\right)\end{aligned}$$

i.e the area of the region is the $\int (\text{top curve} - \text{bottom curve})\,dx$. See example 1-2-4.

Sometimes it's easier to consider x as a function of y. i.e the area of the region bounded by $x = f(y), x = g(y)$ between $y = c$ and $y = d$ where $f(y) \geq g(y)$ for all y in $[c, d]$ is the area $\int_c^d (f(y) - g(y)\,dy$ or $\int (\text{right curve} - \text{left curve})\,dy$.

Example: find the area enclosed by $y = x$ and $y^2 = x + 2$. Points of interseection of $x = y$ and $x = y^2 - 2$

$$\begin{aligned}y &= y^2 - 2\\ y^2 - y - 2 &= 0\\ (y-2)(y+1) &= 0\\ y = -1 &\qquad y = 2\end{aligned}$$

$$\begin{aligned}\text{area} &= \int_{-1}^{2} \left(y - \left(y^2 - 2\right)\right)\,dy\\ &= \int_{-1}^{2} \left(y - y^2 + 2\right)\,dy\\ &= \left(\frac{1}{2}y^2 - \frac{1}{3}y^3 + 2y\right)\Big|_{-1}^{2}\\ &= \left(\frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 + 2(2)\right) - \left(\frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 + 2(-1)\right)\\ &= 2 - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2\\ &= 4\frac{1}{2}\end{aligned}$$

See example 5.

What if we were given a table of values of the function?

x	0	2	4	6	8	10	12
$f(x)$	7	8	7	7	9	8	8
$g(x)$	3	5	4	2	3	4	3
$f(x) - g(x)$	4	3	3	5	6	4	5

Then we could use a numerical integration method, like Simpson's Rule, to approximate the area. Recall Simpson's Rule. p. 422

$$\int_{\alpha}^{\beta} \varphi(x)\, dx \approx \frac{\Delta x}{3} \left[\varphi(x_0) + 4\varphi(x_1) + 2\varphi(x_2) + 4\varphi(x_3) + \ldots + 4\varphi(x_{n-1}) + \varphi(x_n)\right]$$

So here $\Delta x = 2$, $n = 6$, $x_0 = 0$, $x_1 = 2, \ldots, x_6 = 12$ and the funcction is $\varphi(x) = f(x) - g(x)$. So:

$$\begin{aligned}\int_0^{12} (f(x) - g(x))\; dx &\approx \frac{2}{3}\left[4 + 4(3) + 2(3) + 4(5) + 2(6) + 4(4) + 5\right] \\ &= \frac{2}{3}(4 + 12 + 6 + 20 + 12 + 16 + 5) \\ &= \frac{2}{3}(75) \\ &= 50\end{aligned}$$

See also Ex.4. p.434-5

6.2 : Volumes

Consider a solid object lying between $x = a$ and $x = b$. What is its volume? Chop the interval $[a, b]$ into n pieces of width or thickness $\Delta x = \dfrac{b-a}{n}$ on each subinterval $[x_{i-1}, x_i]$, take a sample point x_i^* at x_i^*, look at the cross section area of the solid is a plane perpendicular to the x-axis, call it $A(x_i^*)$ then the volume of the solide on the subinterval is approximately $A(x_i^*)\Delta x$ so the total volume is approximately:

$$\sum_{i=1}^{n} A(x_i^*)\Delta x$$

and the approximation gets better as $n \to \infty$ (or $\Delta x \to \infty$).

So we have : p.439

If S is a solid that lies between $x = a$ and $x = b$ and tthe cross-section area of S in a plane through x and perpendicular to the $x - axis$ is $A(X)$, then the volume of S is:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx$$

Examples:

1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ around the line $y = 3$.

   The cross-sectional area will be washers in rings. The inner radius is $r_{in} = 3-x$ and the outer radius is $r_{out} = 3 - x^2$ so the area of the washer is:

$$\begin{aligned} A(x) &= \pi r_{out}^2 - \pi r_{in}^2 \\ &= \pi \left(3 - x^2\right)^2 - \pi(3-x)^2 \\ &= \pi \left(9 - 6x^2 + x^4 - \left(9 - 6x + x^2\right)\right) \\ &= \pi \left(x^4 - 7x^2 + 6x\right) \end{aligned}$$

$$\begin{aligned} \text{so the volume is } V &= \int_0^1 \pi \left(x^4 - 7x^2 + 6x\right)\, dx \\ &= \pi \left(\frac{1}{5}x^5 - \frac{7}{3}x^3 + 3x^2\right)\Bigg|_0^1 \\ &= \pi \left(\frac{1}{5} - \frac{7}{3} + 3\right) \\ &= \frac{13}{15}\pi \\ &\approx 2.7227 \end{aligned}$$

   See also examples 1-5.

   (Solids of this sort are called *solides of revolution*)

2. What if we had rotated the region around the line $x = 0$? Then we would change our approach slightly and integrate over y. Now, cross-section in planes perpendicular to the y-axis are washers with $r_{\text{in}} = y$ and $r_{\text{out}=\sqrt{y}}$.

   So the cross sectioned area is :

$$A(y) = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2 = \pi(y - y^2)$$

   So the volume is :

$$V = \int_0^1 \pi(y - y^2)\, dy = \pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Bigg|_0^1 = \frac{\pi}{6} \approx 0.5236$$

3. (Example 8) Find the volume of a pyramid of height h with square base with sides s of length L.

   The cross-section of height y is a square.

The lengths of the sides of the square are $2s$ where:

$$\frac{s}{h-y} = \frac{\frac{L}{2}}{h} \qquad \text{by similar triangles}$$
$$s = \frac{L}{2h}(h-y)$$
$$\text{so} \qquad 2s = \frac{L}{h}(h-y) \qquad \text{and the squares}$$
$$\text{has area } A = (2s)^2 = \frac{L^2}{h^2}(h-y)^2$$
$$\text{so } V = \int_0^h \frac{L^2}{h^2}(h-y)^2\,dy$$
$$= \frac{L^2}{h^2}\left[-\frac{1}{3}(h-y)^3\Big|_0^h\right]$$
$$= \frac{L^2 h}{3}$$

See also example 7. p. 444-5

Example : Base is parabolic region $\{(x,y) \mid x^2 \le y \le 1\}$.

Cross-sections perpendicular to y-axis are equilateral triangles. But $x = \sqrt{y}$. Base $2x = 2\sqrt{y}$. Height $h = \sqrt{3}x$.

So the area of the cross-section is:

$$A(y) = \frac{1}{2}bh = \frac{1}{2}2\sqrt{y}\sqrt{3}\sqrt{y} = \sqrt{3}y$$

and so the volume is:

$$V = \int_0^1 A(y)\,dy = \int_0^1 \sqrt{3}y\,dy = \frac{\sqrt{3}}{2}y^2\Big|_0^1 = \frac{\sqrt{3}}{2}$$

6.3 : Volumes by Cylindrical Shells

Suppose we have the region bounded by $y = f(x)$, the x-axis (i.e $y = 0$) and the lines $x = a$ and $x = b$ and we rotate the region around the y-axis.

Shop the interval $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$ and let $\overline{x}_j$ be the midpoint of subinterval $[x_{j-1}, x_j]$ from the rectangle with height $f(x)$ and rotate it around the y-axis to obtain a cylindrical shell with volume:

$$V_j = 2\pi\overline{x}_j f(\overline{x}_j)\Delta x$$

The volume of our solid is then approximately the sum of the volumes of these shells, i.e:

$$V \approx \sum_{j=1}^{n} V_j = \sum_{j=1}^{n} 2\pi \overline{x}_j f(\overline{x}_j)\Delta x$$

Then the limit as $n \to \infty$ (or $\Delta x \to 0$) to get the true value:

$$V = \int_a^b 2\pi x f(x)\, dx$$

Example: find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = x - x^2$ ($y = x(1-x)$) and $y = 0$. See example 2.

$$\begin{aligned} V &= \int_0^1 2\pi x(x - x^2)\, dx \\ &= 2\pi \int_0^1 \left(x^2 - x^3\right)\, dx \\ &= 2\pi \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 \\ &= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \frac{2\pi}{12} = \frac{\pi}{6} \end{aligned}$$

See also example 1-4.

6.4 : Arc Length

Suppose we're interested in the length of the curve $y = f(x)$ from $x = a$ to $x = b$.

We'll chop the curve into little pieces and try to estimate the length of each little piece and then sum over the pieces.

When the pieces are small, they will be approximately straight and they will be the corresponding small changes in x and y, Δx and Δy since the curve is approximately straight, we can use Pythagoras' theorem to approximate its lenght $l \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Now, we need to replace Δy by an expression in x only (because we'll want to integrate over x from a to b) but recall that $f'(x) \approx \dfrac{\Delta y}{\Delta x}$ so $\Delta y \approx f'(x)\Delta x$, so:

$$l \approx \sqrt{(\Delta x)^2 + (f'(x)\Delta x)^2} = \sqrt{1 + (f'(x))^2}\Delta x$$

Now, all we have to do is sum up over all of the little pieces to get the total arc length $L \approx \sum \sqrt{1 + (f'(x))^2}\Delta x$. Now take the limite as $\Delta x \to 0$ to get the formula for the arc length of curve $f(x)$ between $x = a$ and $x = b$:

$$\text{arc length } L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example: calculate the arc length of $y = 4x^{\frac{3}{2}}$ from $x = 1$ to $x = 4$.

$$f(x) = 4x^{\frac{3}{2}} \qquad \Longrightarrow \qquad f'(x) = 6x^{\frac{1}{2}}$$

$$\begin{aligned}
\text{so } L &= \int_1^4 \left(1 + \left(6x^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \, dx \\
&= \int_1^4 (1 + 36x)^{\frac{1}{2}} \, dx \\
&= \left(\frac{2}{3}\right)\left(\frac{1}{36}\right)(1 + 36x)^{\frac{3}{2}}\Big|_1^4 \\
&= \frac{1}{54}\left((145)^{\frac{3}{2}} - (37)^{\frac{3}{2}}\right) \\
&\approx 28.17
\end{aligned}$$

Note: even though the formula for arc length is simple, it usually leads to integrals that have to be done numerically (as antiderivatives can't be found easily - or at all). See example 2. p.457

If the curve is given as $x = g(y)$, $c \leq y \leq d$, then a similar derivation will lead to $L = \displaystyle\int_c^d \sqrt{(g'(y))^2 + 1}\, dy$. See example 3. p.457-8

Another way that a curve can be described is parametrically as $x = f(t)$ and $y = g(t)$ where the parameter t is specified over some interval. See example 1-4. p.171-6 p.72-6

If the curve is specified by the parametric equation $x = f(t), y = g(t)$ for $\alpha \leq t \leq \beta$, then:

$$l \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\Delta t \approx \sqrt{(f'(t))^2 + (g'(t))^2}\Delta t$$

So: p.456

$$L = \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Can you see how the previous cases are just special cases of this?

Example: find the length of the curve given by $x = t^3$, $y = 2t^2$ from the points $(0, 0)$ to $(8, 8)$.

$$
\begin{aligned}
\text{Arc Length } L &= \int_0^2 \sqrt{\left(3t^2\right)^2 + (4t)^2}\, dt \\
&= \int_0^2 \sqrt{9t^4 + 16t^2}\, dt \\
&= \int_0^2 t\sqrt{9t^2 + 16}\, dt \\
&= \left(\frac{2}{3}\right)\left(\frac{1}{18}\right)\left(9t^2 + 16\right)^{\frac{3}{2}}\Big|_0^2 \\
&= \frac{1}{27}\left(9t^2 + 16\right)\Big|_0^2 \\
&= \frac{1}{27}\left[(36+16)^{\frac{3}{2}} - (16)^{\frac{3}{2}}\right] \\
&\approx 11.518
\end{aligned}
$$

See also examples 1 and 4.

6.5 : Average Value of a Function

What is the average value of a function $y = f(x)$ over an interval $a \leq x \leq b$?

Chop the interval $[a, b]$ into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$. In each subinterval $[x_{i-1}, x_i]$, take a sample point x_i^*, then the average value can be approximated by:

$$
f_{\text{ave}} \approx \frac{f\left(x_1^*\right) + f\left(x_2^*\right) + \ldots + f\left(x_n^*\right)}{n} = \frac{\sum_{i=1}^n f(x_i^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}\sum_{i=1}^n f(x_i^*)\Delta x
$$

As $n \to \infty$, we'll get the true value, i.e

$$
f_{\text{ave}} = \lim_{n\to\infty} \frac{1}{b-a} \underbrace{\sum_{i=1}^n f(x_i^*)\Delta x}_{\text{Riemann sum}} = \frac{1}{b-a}\int_a^b f(x)\, dx
$$

Example: find the average value of $f(x) = \sin x$ on $[0, \pi]$.

$$
\begin{aligned}
f_{\text{ave}} &= \frac{1}{\pi - 0}\int_0^\pi \sin x\, dx \\
&= \frac{1}{\pi}\left(-\cos x|_0^\pi\right) \\
&= -\frac{1}{\pi}(-1-1) \\
&= \frac{2}{\pi} \\
&\approx 0.6366
\end{aligned}
$$

See example 1 and 3.

Notice that since $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$, we'll also have $\int_a^b f(x)\,dx = f_{\text{ave}}(b-a)$, i.e the area under the curve is equal to the average value times the length of the interval. p.462

We could also notice that there are two values of x, say x_1 and x_2 where:

$$f(x_1) = f(x_2) = \frac{2}{\pi} = f_{\text{ave}}$$
$$\sin(x_1) = \frac{2}{\pi} \Longrightarrow x_1 = \arcsin\left(\frac{2}{\pi}\right) \approx 0.6901 \ (\approx 39.54^\circ)$$
$$\text{and } x_2 = \pi - x_1 \approx 2.4515 \ (\approx 140.46^\circ)$$

Is this a coincidence? No.

The Mean Value Theorem for Integrals p.462

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that :

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Or $\int_a^b f(x)\,dx = f(c)(b-a)$. See also example 2. p.462

6.6 : Applications to Physics and Engineering

Work (p.464-7).

If an object is moved a distance d against a constant force F then the work done is $W = Fd$. But if the force varies with distance, then we have to chop the path of the object up into pieces over which the force is approximately constant and sum over the pieces. More specifically, if we are moving the object along the x-axis from $x = a$ to $x = b$ and the force is $f(x)$, then chop $[a, b]$ into n subintervals of length Δx, take sample point x_i^* from ith subintervals $[x_{i-1}, x_i]$ and the force is approximately $f(x_i^*)$ over that subinterval, so the work on the subinterval is:

$$W_i \approx f(x_i^*)\Delta x$$

then the total work is :

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

and then take the limit :

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

Be careful about the difference between mass m and weigth mg and with the units. See example 1. p.465
p.466

Example : (p.472 # 6, Hooke's Law)

A spring has a natural length 20 cm. If a 25 N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 to 25 cm ?

Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural/equilibrium length is $f(x) = kx$ where k is the *spring constant*.

30 cm is 10 cm beyond equilibrium, so $x = 10$ cm $= 0.1$ m, so 25 N $= k(0.1 \text{ m}) \implies k = 250$ N/m and $f(x) = 250x$ and so the work required to stretch 5 cm = 0.05 m beyond equilibrium is:

$$W = \int_0^{0.05} f(x)\,dx = \int_0^{0.05} 250x\,dx = 125x^2\Big|_0^{0.05} = 0.3125\text{J (or Nm)}$$

See also example 2.

Example : a tank is filled with water and has the shape of an isosceles trapezoid (at the ends) and is 1 m wide and 1.5 m high.

How much work is required to pump all of the water out? ($\rho = 1000\ \text{kg/m}^3$). The work required to pump the water out is the same as that required to ligt all the water to the top. Consider a horizontal slice of the water at depth x_i of thickness Δx. The volume of this slice is $V_i = l_i \Delta x(1)$.

$$\begin{aligned}
l_1 &= 1 + 2a \text{ (m)} \\
\text{by similar triangles } \frac{a}{0.5} &= \frac{1.5 - x_i}{1.5} \\
\text{so } a &= 0.5\frac{(1.5 - x_i)}{1.5} \\
\text{so } l_i &= 1 + \frac{1.5 - x}{1.5} \\
&= \frac{1.5 + 1.5 - x}{1.5} \\
&= 2 - \frac{x}{1.5} \\
&= 2 - \frac{x}{\frac{3}{2}} \\
\text{and } Vi &= \left(\frac{3 - x_i}{1.5}\right)\Delta x \\
&= \left(2 - \frac{2}{3}x_i\right)\Delta x
\end{aligned}$$

The mass of water in the slice is $m_i = \rho V_i = 1000\left(2 - \frac{2}{3}x_i\right)\Delta x$ and then the work done to lift this slice to the top is:

$$\begin{aligned}
W_i &= \underbrace{m_i g}_{F_i}\, x_i \\
&= \left(1000\left(2-\frac{2}{3}x_i\right)\Delta x\right)(9.8)x_i \\
&= 9800\left(2x_i - \frac{2}{3}x_i^2\right)\Delta x \\
\text{so the work done is } W &= \lim_{n\to\infty}\sum_{i=1}^{n} W_i \\
&= \int_0^{1.5} 9800\left(2x-\frac{2}{3}x^2\right)\,dx \\
&= 9800\left(x^2-\frac{2}{9}x^3\right)\Big|_0^{1.5} \\
&= 14700\text{ J}
\end{aligned}$$

See also example 3 and 4. p.466-7

Hydrostatic Pressure and Forces. p.474-5

We'll look at the force exerted by water on dams or containers. The more fundamental concept is pressure, which is the force per unit area exerted by the water. Some important things to know about pressure in any fluid, gas or liquid, are:

- Pressure is exerted equally in *all* directions,
- Pressure increases with depth (which is why the air is thinner at higher altitudes)

But we'll stick with liquids, typically water. If the liquid has density ρ, the pressure at depth h will be $P = \rho g h$ and since pressure is force/area, $F = P\cdot$ Area.

If the pressure is not constant, we'll do our usual trick and divide the surface into pieces over which the pressure is approximately constant.

Example: a lobster tank in a restaurant is 1.5 m long by 1 m wide by 0.7 m deep. Find the water force on the bottom and on each of the four sides ($\rho = 1000$ kg/m^3).

On the sides:

Take a horizontal strip at height h with width Δh. The area of this strip will be $1.5\Delta h$ m^2. The pressure at this height will be:

$$P = \rho\cdot g\cdot \text{depth} = \rho g(0.7-h)$$

So the force on the strip is $F = 1.5\rho g(0.7-h)\Delta h$ so that total force is $F = \sum 1.5\rho g(0.7-h)\Delta h$. Taking the limit as $\Delta h \to 0$ gives:

$$\begin{aligned} F &= \int_0^{0.7} (1.5 \text{ m}) \left(1000 \text{ kg/m}^3\right) \left(9.8 \text{ m/s}^2\right) (0.7 - h)\, dh \\ &= 14700 \int_0^{0.7} (0.7 - h) dh \\ &= 14700 \left[0.7h - \frac{1}{2}h^2\right]\Big|_0^{0.7} \\ &= 14700 \left((0.7)(0.7) - \frac{1}{2}(0.7)(0.7)\right) \\ &= 3601.5 \text{ N} \end{aligned}$$

On the ends:

Now the horizontal strip has area Δh m^2 and the pressure is $P = \rho g(0.7 - h)$ so the force is $F = \rho g(0.7 - h)\, dh$, so the integral is:

$$\begin{aligned} F &= \int_0^{0.7} (1000)(9.8)(0.7 - h)\, dh \\ &= 9800 \left((0.7)^2 - \frac{1}{2}(0.7)^2\right) \\ &= 2401 \text{ N} \end{aligned}$$

See also example 5.

<u>Moments and Centres of Mass.</u> (p. 469-72)

If we have a system of n particles of masses $m_1, m_2, \ldots, m_n$ and located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then the *centre of mass* of the system is located at the point $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}, \overline{y} = \dfrac{M_x}{m}$ where $m = \sum_{i=1}^{n} m_i$ is the total mass and $M_y = \sum_{i=1}^{n} m_i x_i$ is the *moment about the y-axis* (tendency to rotate about the y-axis) and $M_x = \sum_{i=1}^{n} m_i y_i$ is the *moment about the x-axis* (rotate around x). See pages 469-70 and example 6.

Now if we have a *lamina* (a flat object) with constant density ρ that lies beneath the graph of $y = f(x)$ between $x = a$ and $x = b$, then the moments are $M_y = \rho \int_a^b x f(x)\, dx$ and $M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\, dx$.

The mass of the region is $m = \rho A = \rho \int_a^b f(x)\, dx$ and so the location of the centre of mass or *centroid* is:

$$\begin{aligned} \overline{x} = \frac{M_y}{m} &= \frac{\rho \int_a^b x f(x)\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx} = \frac{1}{A}\int_a^b x f(x)\, dx \\ \overline{y} = \frac{M_x}{m} &= \frac{\rho \int_a^b \frac{1}{2}(f(x))^2\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b \frac{1}{2}(f(x))^2\, dx}{\int_a^b f(x)\, dx} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\, dx \end{aligned}$$

Example: (modified p.456 # 48) find the moments M_y and M_x and location of the centroid of the region bounded by $y = \frac{1}{x}, y = 0, z =, x = 2$ (assume the density is $\rho = 1$).

$$\begin{aligned}
M_y &= \rho \int_a^b x f(x)\, dx \\
&= (1) \int_1^2 x \left(\frac{1}{x}\right)\, dx \\
&= \int_1^2 dx = 2 - 1 \\
&= 1 \\
M_x &= \rho \int_a^b \frac{1}{2} \left(f(x)\right)^2\, dx \\
&= (1) \int_1^2 \frac{1}{2} \left(\frac{1}{x}\right)^2\, dx \\
&= \frac{1}{2} \int_1^2 \frac{1}{x^2}\, dx \\
&= \frac{1}{2}\, -\frac{1}{x}\bigg|_1^2 \\
&= -\frac{1}{2} \left(\frac{1}{2} - 1\right) \\
&= \frac{1}{4}
\end{aligned}$$

The area is:

$$A = \int_a^b f(x)\, dx = \int_1^2 \frac{1}{x}\, dx = \ln x|_1^2 = \ln 2 - \ln 1 = \ln 2$$

Mass $m = \rho A = A$. So the centroid is:

$$\begin{aligned}
(\overline{x}, \overline{y}) &= \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \\
&= \left(\frac{1}{((1)\ln 2)}, \frac{\frac{1}{4}}{((1)\ln 2)}\right) \\
&= \left(\frac{1}{\ln 2}, \frac{1}{4 \ln 2}\right) \\
&\approx (1.44, 0.36)
\end{aligned}$$

See also example 7. p. 472

Chapter 7 : Differential Equations

7.1 : Modelling With Differential Equations

We've been solving differential equations already - when we antidifferentiate to find $\int f(x)\,dx = F(x) + C$ (when $F'(x) = f(x)$). We've been solving the differential equation:

$$\frac{dy}{dx} = f(x)$$

Just to show that this is so:

$$\frac{dy}{dx} = f(x) \qquad \Longrightarrow \qquad dy = f(x)dx$$

Now integrate on both sides:

$$y = \int dy = \int f(x)\,dx = F(x) + C$$

Definition: A *differential equation* is an equation involving an *unknown* function $y(x)$ and its derivatives and the independant variable x. Example :

$$\frac{dy}{dx} = y + x, \qquad y' = \frac{1}{2}x + 3, \qquad y'' + 3y' + y = x$$

The *order* of a differential equation is the order of the highest-order derivative appearing in the equation, so $y' = x + 7$, the order is 1, but $y'' + y = 0$ has order 2. The *solution* to a differential equation is the *function* $y(x)$. Example : the differential equation $y' = x + 7$ has solution $y(x) = \frac{1}{2}x^2 + 7x + C$. To show that this is so, just substitute:

$$\begin{aligned} \text{if } y = \frac{1}{2}x^2 + 7x + C, \text{ then } \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{2}x^2 + 7x + C\right) \\ &= x + 7 \end{aligned}$$

Note that this solution contains an *arbitraty constant* C. From the antidifferentiation:

$$\begin{aligned} &\text{if } \frac{dy}{dx} = x + 7 \\ &\text{then } y = \int dy = \int (x + 7)\,dx = \frac{1}{2}x^2 + 7x + C \end{aligned}$$

What does this really mean? It means that there are *infinitely many solutions* to the diff. eq. $y' = x+7$ - one for each value of C. For this reasion, $y = \frac{1}{2}x^2 + 7x + C$ is called the *general solution* because it represents the whole *family of solution* - ie the whole family of curves that have slope $x + 7$.

In order to know the value of C, we need to have an *initial condition* $y(x_0) = y_0$ (essentially, this gives us a point that the curve must pass through - and only one member of the family will do this).

So say we had been given $y' = x + 7, y(0) = 3$ (a diff. eq. with an initial condition is called an *initial-value problem*). We know the general solution is $y(x) = \frac{1}{2}x^2 + 7x + C$ but we need to satisfy $y(0) = 3$, so $y(0) = \frac{1}{2}(0)^2 + 7(0) + C = 3 \implies C = 3$. So the solution that satisfies the initial condition is:

$$y(x) = \frac{1}{2}x^2 + 7x + 3$$

and this is called a *particular* or *unique solution* (it is the *unique* curve that has slope $x + 7$ and passes through $(0, 3)$).

Where do differential equation come from? Often when we are trying to model physical phenomena, we do not know the function that describes the system's behavior, but we can observe the behavior of its rate (see the discussion and example on p.494-99).

7.2 : Direction Fields and Euler's Method

We'll look at a graphical method of solving first-order differential equation.

Idea: if we're trying to solve $y' = f(x, y)$, then we know that at the point (x, y), the slope of the solution curve is $f(x, y)$ - i.e we know it so we can draw a *(slope) direction field* which is a graph of the general begavior of y'.

At each point (x, y), calculate $y' = f(x, y)$ and draw a small line segment that indicates this slope. Do this for many points to produce the slope field.

See the examples on p.500-3.

Euler's Method

This is a numerical method that approximates the solution of the differential equation. Essentially, what the slope field did "graphically", Euler's Method will do "numerically".

How it works: say we have differential equation $y' = f(x, y)$ and initial condition $y(x_0) = y_0$. From what we've just done with slope fields, we know that y' gives us a direction to head in. So take small step (Δh), the corresponding change in y is $\Delta y = f(x_0, y_0)(\Delta h)$ (since $f(x, y) = y' \approx \frac{\Delta y}{\Delta x}$)). So our new y value, call it y_1 is:

$$y_1 = y_0 + \Delta y = y_0 + f(x_0, y_0)(\Delta h)$$

And this is the value that corresponds with $x_1 = x_0 + (\Delta h)$. At our new point, (x, y), we have slope $f(x_1, y_1)$ so if we take another step of (Δh), $\Delta y = f(x_1, y_1)(\Delta h)$, and so :

$$y_2 = y_1 + f(x_1, y_1)(\Delta h)$$

You keep doing this for as many steps as required/desired and you generate a set of points (x_0, y_0), (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) where y_j is a numerical approximation of the *true* or *exact* value of $y(x_j)$.

The general formula for Euler's Method is:

$$y_{n+1} = y_n + f(x_n, y_n)(\Delta h) \tag{1}$$

The *error* of Euler's Method is the difference between the calculated approximation and the exact value. i.e:

$$\text{error} = y(x_j) - y_j$$

and if the number of steps used is n, the error will be approximately proportional to $\frac{1}{n}$.

Reason : the longer n is, the smaller (Δh) will be for a given problem.

Example : use then steps of Euler's Method to approximate the solution of $y' = \frac{1}{x}, y(1) = 0$ with step size $h = 0.1$

$$y(1) = 0 \quad \Longrightarrow \quad x_0 = 1, y_0 = 0$$
$$\text{then } x_1 = 1.1, x_2 = 1.2, \text{etc}\ldots$$
$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + \frac{0.1}{x_n}$$

$$\begin{aligned}
\text{so } y_1 &= y_0 + \frac{0.1}{x_0} = 0 + \frac{0.1}{1} = 0.1 \\
y_2 &= y_1 + \frac{0.1}{x_1} = 0.1 + \frac{0.1}{1.1} = 0.190909 \\
y_3 &= y_2 + \frac{0.1}{x_2} = 0.190909 + \frac{0.1}{1.2} = 0.274242 \\
y_4 &= y_3 + \frac{0.1}{x_3} = 0.274242 + \frac{0.1}{1.3} = 0.351165 \\
y_5 &= y_4 + \frac{0.1}{x_4} = 0.351165 + \frac{0.1}{1.4} = 0.422594 \\
y_6 &= y_5 + \frac{0.1}{x_5} = 0.422594 + \frac{0.1}{1.5} = 0.489261 \\
y_7 &= y_6 + \frac{0.1}{x_6} = 0.489261 + \frac{0.1}{1.6} = 0.551761 \\
y_8 &= y_7 + \frac{0.1}{x_7} = 0.551761 + \frac{0.1}{1.7} = 0.610585 \\
y_9 &= y_8 + \frac{0.1}{x_8} = 0.610585 + \frac{0.1}{1.8} = 0.666141 \\
y_{10} &= y_9 + \frac{0.1}{x_9} = 0.666141 + \frac{0.1}{1.9} = 0.718773
\end{aligned}$$

Exercise - compare with the true solution $y = \ln x$. See the example on pages 504-5. **p. 504-5**

7.3 : Separatable Equations

A D.E is called *separable* if it can be written in the form:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

because we can then separate the variables as $g(y)dy = f(x)dx$ (y on one side, x on the other).

To solve, all we need to do is integrate on both sides:

$$\int g(y)dy = \int f(x)dx + C$$

If $G(Y)$ and $F(x)$ are antiderivatives of $g(y)$ and $f(x)$, we'll have $G(y) = F(x) + C$ (only need one constant).

Examples:

1. $\frac{dP}{dt} = kP$ or $\frac{dP}{P} = kdt$ so $\int \frac{dP}{P} = \int kdt + C$.
   So $\ln|P| = kt + C$ or $P(t) = e^{kt+C} = e^C e^{kt} = Qe^{kt}$ ($Q = P_0$)

2. $\frac{dy}{dx} = xy^2 \cos(x^2) \qquad y(0) = 1$. Separate to get:

$$\frac{dy}{y^2} = x\cos(x^2)dx$$
$$\int \frac{dy}{y^2} = \int x\cos(x^2)dx + C$$
$$-\frac{1}{y} = \frac{1}{2}\sin(x^2) + C$$

   So the general solution is $y = \frac{-2}{\sin(x^2) + K}$.

   $y(0) = 1 \quad \Longrightarrow \quad K = -2$ so the unique solution is $y = \frac{-2}{\sin(x^2) - 2} = \frac{2}{2 - \sin(x^2)}$. See also examples 1-4.

Example: the velocity of a falling body.

Think of an object falling from a great height. We know that gravity will act on the object, causing it to accelerate.

The force of gravity on the object is mg (we'll take downwards as positive - then velocity will be positive) but air resistance will also act on the object, causing it to slow down.

Like all frictional forces, air resistance, at least to a good approximation, is proportional to velocity (i.e the faster the object goes, the more strongly the resistance).

So the force of air resistance will be of the form $-kv$ for proportionality constance $k > 0$ (recall that upwards is negative).

So the total force on this body is $F = \text{gravity} + \text{air resistance} = mg - kv$.

By Newton's 2nd Law, $F = ma$ but $a = \frac{dv}{dt}$, so we have $mg - kv = m\frac{dv}{dt}$ (a differential equation in valocity v).

Reunite as :

$$\frac{dv}{dt} = g - \frac{k}{m}v = -\frac{k}{m}\left(v - \frac{m}{k}g\right)$$

Separate the variables :

$$\frac{dv}{v - \frac{mg}{k}} = -\frac{k}{m}dt$$

Integrate:

$$\int \frac{dv}{v - \frac{mg}{k}} = \int -\frac{k}{m}dt$$

To get :

$$\ln\left|v - \frac{mg}{k}\right| = -\frac{k}{m}t + C$$

Exponentiate both sides to get:

$$\begin{aligned}\left|v - \frac{mg}{k}\right| &= e^{\frac{-kt}{m}+C} \\ &= e^{C}e^{\frac{-kt}{m}} \\ &= Ke^{\frac{-kt}{m}}\end{aligned}$$

Which is $v - \frac{mg}{k} = Ae^{\frac{-kt}{m}}$ (general solution).

A is determined by the initial conditions. we'll assume that $v(0) = 0$, i.e the object starts at rest at time 0, then:

$$0 - \frac{mg}{k} = Ae^{0} \qquad \Longrightarrow \qquad A = -\frac{mg}{k}$$

So our particular solution is $v - \frac{mg}{k} = -\frac{mg}{k}e^{-\frac{kt}{m}}$ or $v = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$.

Notice that, as $t \to \infty$, $v \to \frac{mg}{k}$. The solution looks like:

So this means that there is a maximum velocity that the object can attain, $v = \frac{mg}{k}$ (though in this model it is never actually reached).

$\frac{mg}{k}$ is called the *terminal velocity*.

How does it come about? Recall that $F = mg - kv$ - we can have the situation when gravity has accelerated the object to such a velocity v that air resistance will balance gravity and then total force is zero, i.e $F = mg - kv = 0 \quad \Longrightarrow \quad v = \frac{mg}{k}$. But if $F = 0$, $\frac{dv}{dt} = 0$ and the object's velocity is no longer changing.

So in conclusion, if the object falls from a point high enough (so that it falls for a long enough time), it will accelerate up to its terminal velocity $\frac{mg}{k}$ and then fall at that constant speed.

Example: salt concentration.

Suppose there is a reservoir holding 100 million litres of water (i.e 100 Ml) that supplies a city with 1 million litres per day.

The reservoir is fed by two sources - a spring which provides 0.9 million litres per day and run-off from the surrounding land which provide 0.1 million litres per day.

The spring is clean, but the run-off water contains salt with a concentration of 0.0001 kg per litre (it's winter time and salt is being used on the roads). Assume the following :

1. There is no salt in the reservoir initially.

2. The reservoir is well-mixed.

3. The reservoir is strip full.

Find the concentration of salt in the reservoir as a function of time.

First of all, concentration is quantity/volume, but we'll work with the quantity Q. At all time, we have sand entering through the run-off water and leaving through consumption by the city, so rate of change of Q = rate salt entering − rate salt leaving. So we need to work out these two rates on the RHS:

- rate salt entering :

  This is easy because it's constant.

$$\begin{aligned}\text{rate salte entering } &= \text{concentration of salt in run-off water}\\ &\quad \times \text{ volume of run-off water entering reservoir per day}\\ &=0.0001\frac{\text{kg}}{l}\times 100000\frac{l}{\text{day}}\\ &=10\frac{\text{kg}}{\text{day}}\end{aligned}$$

- rate salt leaving :

The city uses water of concentration Q / 100 Ml, so :

$$\begin{aligned} \text{rate salt leaving} &= \text{concentration of salt in reservoir} \\ &\quad \times \text{volume of water used by city per day} \\ &= \frac{Q}{100\ \text{M}l} \times 1 \frac{\text{M}l}{\text{day}} \\ &= \frac{Q}{100} \frac{\text{kg}}{\text{day}} \end{aligned}$$

$\therefore$ the differential equation for Q is:

$$\begin{aligned} \frac{dQ}{dt} &= \text{rate of change of } Q \\ &= \text{rate entring } - \text{ rate leaving} \\ &= 10 - \frac{Q}{100} \end{aligned}$$

so $\dfrac{dQ}{dt} = 10 - \dfrac{Q}{100} = -\dfrac{1}{100}(Q - 1000) = -0.01(Q - 1000)$

Seperate variables :

$$\frac{dQ}{Q - 1000} = -0.01dt$$

Integrate :

$$\int \frac{dQ}{Q - 1000} = \int -0.01dt$$

To get :

$$\ln |Q - 1000| = -0.01t + C$$

So the general solution is $Q - 1000 = Ae^{-0.01t}$ and we know that $Q(0) = 0$ so:

$$0 - 1000 = Ae^0 \qquad \Longrightarrow \qquad A = -1000$$

So the particular solution is $Q = 1000\left(1 - e^{-0.01t}\right)$(kg) and therefore, the concentration is :

$$C = \frac{Q}{100\ \text{M}l} = \frac{1000\left(1 - e^{-0.01t}\right)\ \text{kg}}{100 \times 10^6\ l} = 10^{-5}\left(1 - e^{-0.01t}\right)\ \text{kg}/l$$

Again, our solution will have a horizontal asymptote.

See also example 6. p.513

7.4 : Exponential Growth and Decay

The solution of the DE $\dfrac{dy}{dx} = ky$ is $y = Ae^{kt}$ (we have done $\dfrac{dP}{dt} = kP$), where $A = y(0)$.

If $k > 0$, this represents *exponential growth*. p.520

If $k < 0$, this represents *exponential decay*.

For population P, we'll have DE $\frac{dP}{dt} = kP$, then $P(t) = P_0 e^{kt}$ ($P_0 = P(0)$). Note that we can write $\frac{1}{P}\frac{dP}{dt} = k$.

For this reason, we call k the *relative growth rate*.

Example: if a population grow at a rate of 3% per year:

- $k = 0.03$ and $P(t) = P_0 e^{0.03t}$.

  If the initial population is $P(0) = P_0 = 1000$, we'll have $P(t) = 1000e^{0.03t}$. What is the population 10 years later ?

  $$P(10) = 1000e^{0.03(10)} = 1000e^{0.3} \approx 1350$$

- If $\frac{dP}{dt} = kP$, $P(0) = 250$ and $P(5) = 400$. Then:

$$\begin{aligned}
P(t) &= P(0)e^{kt} = 250e^{kt} \\
P(5) &= 250e^{5k} = 400 \\
e^{5k} &= \frac{400}{250} = 1.6 \\
\text{so } 5k &= \ln(1.6) = 0.47 \\
\text{so } k &= \frac{1}{5}(0.47) = 0.094 \\
\therefore P(t)&250e^{0.094t}
\end{aligned}$$

See also examples 1 and 2 on pages 521-3.

What if we are talking about a radioactive element? Then the mass decays exponentially ($k < 0$), i.e:

$$\frac{dm}{dt} = km$$

So $m(t) = m_0 e^{kt}$.

It is usual to talk about the *half-life* ($t_{\frac{1}{2}}$) of the element, i.e the time required to reduce the amount present by $\frac{1}{2}$:

$$m(t_{\frac{1}{2}}) = \frac{1}{2}m_0 \qquad \Longrightarrow \qquad e^{kt_{\frac{1}{2}}} = 0.5$$

$$kt_{\frac{1}{2}} = \ln(0.5) = -\ln 2$$

$$\text{(relationship between } k \text{ and } t_{\frac{1}{2}}) \qquad t_{\frac{1}{2}} = -\frac{1}{k}\ln 2$$

$$\text{or } k = -\frac{1}{t_{\frac{1}{2}}}\ln 2$$

Example : ${}^{14}_{6}C$ has a half-life of 5730 years. Carbon dating works by comparing the fraction of ${}^{14}_{6}C$ left in a "dead" thing to the amount that is present in living things (which is constantly replenished).

A fossil was found to have 15% ${}^{14}_{6}C$ left, how old is it?

$$m(t) = m_0 e^{kt} = m_0 e^{t(-\frac{\ln 2}{t_{\frac{1}{2}}})} = m_0 e^{\frac{-t\ln 2}{5730}}$$

So if:

$$\frac{m(t)}{m_0} = 0.15 \Longrightarrow e^{-t\frac{\ln 2}{5730}} = 0.15$$

or :

$$-\frac{t\ln 2}{5730} = \ln(0.15) = -1.89712$$

Then :

$$t = \frac{(1.89712)(5730)}{(\ln 2)} \approx 15683 \text{ years}$$

See also example 3 on page 523-4. p. 523-4

Newton's Law of Heating and Cooling. (page 524-5)

The temperature of an object increases or decreases at a rate that is proportional to the difference between its temperature and the temperature of its surroundings (and the constant of proportionality depends on the materials comprising the object in question).

Say that we have a cup of coffee initially at 40° C and a can of coke originally at 8° C and we leave them in a room with temperature 21° C.

What happens ?

well obsviously the coffee cools and the coke warms, but let's look at the details.

For the coffee :

Let $H(t)$ represent the temperature of the coffee, so $H(0) = 40$. The coffee cools, so $\frac{dH}{dt} < 0$. The temperature difference $H - 21 > 0$, so we'll write :

$$\frac{dH}{dt} = -\alpha(H - 21) \qquad (\text{for } \alpha > 0)$$

Separate the variables :

$$\frac{dH}{H - 21} = -\alpha dt$$

Integrate both sides :

$$\int \frac{dH}{H - 21} = \int -\alpha dt$$

And we get :

$$\ln|H - 21| = -\alpha t + C$$

Which we can reunite as $H(t) = Ke^{-\alpha t}$ (why ?) and the general solution is :

$$H(T) = Ke^{-\alpha t} + 21$$

But $H(0) = 40°$ C, so $K = 19$ (why ?) and the particular solution is :

$$H(t) = 21 + 19e^{-\alpha t}$$

For the coke :

Let $C(t)$ represent the temperature of the coke, so $C(0) = 8°$ C. using reasoning similar to that above, we'll have :

$$\frac{dC}{dt} = \beta(21 - C) \qquad \text{(for } \beta > 0\text{)} \qquad \text{(can you explain this?)}$$

Which has a general solution : $C(t) = 21 - Ke^{-\beta t}$ (why?)

and particular solution : $C(t) = 21 - 13e^{-\beta t}$ (why?)

If we plot these solutions, we have :

(remember that $\alpha, \beta > 0$)

So we see that as $t \to \infty$, $H(t) \to 21$ and $C(t) \to 21$ (expected).

Notice that in both cases, we were solving equations of the form $\frac{dT}{dt} = -k(T-21)$, which have solutions of the form $T = 21 + Ae^{-kt}$. Notice also that $T(t) = 21$ (constant) is also a solution to the differential equation (where $A = 0$ - i.e if something starts at the temperature of the room, it will stay at that temperature).

This solution is called the *equilibrium solution* and can be found solving for $\frac{dT}{dt} = 0$, and the equlibrium is called *stable*, because all other solutions tend towards it as $t \to \infty$.

See also example 4.

Compounded Interest. (p.526-7)

If an investment pays interest at a rate of r and the interest is compounded n times per year, then often t years, the value is $A_0\left(1 + \frac{r}{n}\right)^{nt}$, where A_0 is the amount of the original investment.

Example : if we invest 1000 \$ at 5 % interest, then after one year, it will be worth :

$\$1000(1.05)$	=	$\$1050$	annual compounding
$\$1000(1.025)^2$	=	$\$1050.63$	semiannual compounding
$\$1000(1.0125)^4$	=	$\$1050.95$	quaterly compounding
$\$1000(1.004167)^{12}$	=	$\$1051.16$	monthly compounding
$\$1000(1.00096154)^{52}$	=	$\$1051.25$	weekly compounding
$\$1000(1.000136986)^{365}$	=	$\$1051.27$	daily compounding

What if the interest were compounded continualy ? Then we'd have to take the limit as $n \to \infty$:

$$\begin{aligned} \lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} &= A_0 \lim_{n\to\infty} \left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{rt} \\ &= A_0 \lim_{n\to\infty} \left(\left(1 + \frac{1}{m}\right)^{m}\right)^{rt} \\ &= A_0 e^{rt} \end{aligned}$$

Then we'd have $\$1000e^{0.05} = \$1051.27$ (i.e daily interest can be modelled this way).

See example 5. 526-7