MAT2122 Multivariable Calculus (Fall 2014)

Assignment 5 solutions

7.1.12a (3 points)

Evaluate the integral I of the function $f(x,y,z) = x\cos z$ along the path $\mathbf{c} : t \mapsto \left(t, t^2, 0\right)$, $t \in [0,1]$.

Solution:

$$I = \int_0^1 f(x(t), y(t), z(t)) \cdot \|(x'(t), y'(t), z'(t))\| \, dt$$
$$= \int_0^1 t\sqrt{1+4t^2}\, dt = \frac{1}{8}\int_0^1 \sqrt{1+4t^2}\, d\left(1+4t^2\right) = \frac{1}{8}\cdot\frac{2}{3}\left(1+4t^2\right)^{3/2}\Big|_{t=0}^{t=1} = \frac{5\sqrt{5}-1}{12} .$$

7.1.12b (3 points)

Evaluate the integral I of the function $f(x,y,z) = (x+y)/(y+z)$ along the path $\mathbf{c} : t \mapsto \left(t, \frac{2}{3}t^{3/2}, t\right)$, $t \in [1,2]$.

Solution:

$$I = \int_1^2 f(x(t), y(t), z(t)) \cdot \|(x'(t), y'(t), z'(t))\| \, dt$$
$$= \int_1^2 \sqrt{2+t}\, dt = \frac{2}{3}(2+t)^{3/2}\Big|_{t=1}^{t=2} = \frac{2}{3}(8-3\sqrt{3}) .$$

7.2.4a (3 points)

Find the line integral $I = \int_{\mathbf{c}} \omega$, where $\omega = x\,dy - y\,dx$, and $\mathbf{c}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$

Solution:

$$I = \int_0^{2\pi} \omega(\mathbf{c}(t)) \cdot (x'(t), y'(t))\, dt$$
$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = 2\pi .$$

7.2.4b (3 points)

Find the line integral $I = \int_{\mathbf{c}} \omega$, where $\omega = x\,dx + y\,dy$, and $\mathbf{c}(t) = (\cos \pi t, \sin \pi t)$ for $t \in [0, 2]$.

Solution:

$$I = \int_0^{2} \omega(\mathbf{c}(t)) \cdot (x'(t), y'(t))\, dt$$
$$= \int_0^{2} (\cos \pi t, \sin \pi t) \cdot (-\pi \sin \pi t, \pi \cos \pi t)\, dt = 0 .$$

7.2.4c (3 points)

Find the line integral $I = \int_{\mathbf{c}} \omega$, where $\omega = yz\,dx + xz\,dy + xy\,dz$, and $\mathbf{c}$ consists of straight line segments joining $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$.

Solution:

$$\int_{\mathbf{c}} \omega = \int_{\mathbf{c}_1} \omega + \int_{\mathbf{c}_2} \omega \ ,$$

where $\mathbf{c}_1(t) = (1-t, t, 0)$ and $\mathbf{c}_2(t) = (0, 1-t, t)$ for $t \in [0,1]$. Each of the integrals in the right-hand side is equal to 0: the first one, because the curve $\mathbf{c}_1$ is contained in the plane $\{z = 0\}$, whereas the coefficients in front of dx and dy in ω are multiples of z; in the same way, $\mathbf{c}_2$ is contained in the plane $\{x = 0\}$, whereas the coefficients in front of dy and dz in ω are multiples of x. Thus, $I = 0$.

7.2.4d (3 points)

Find the line integral $I = \int_{\mathbf{c}} \omega$, where $\omega = x^2\,dx - xy\,dy + dz$, and $\mathbf{c}(x) = \left(x, 0, x^2\right)$ for $x \in [-1, 1]$

Solution:

$$I = \int_{-1}^{1} \left(x^2, 0, 1\right) \cdot (1, 0, 2x)\,dx = \int_{-1}^{1} \left(x^2 + 2x\right)\,dx = \left(\frac{1}{3}x^3 + x^2\right)\Bigg|_{x=-1}^{x=1} = \frac{2}{3} \ .$$

7.2.14 (3 points)

Show that the integral I of the vector field $\mathbf{F} = \left(z^3 + 2xy, x^2, 3xz^2\right)$ around the circumference of the square with vertices $(\pm 1, \pm 1)$ is zero.

Solution: The integral I can be decomposed as $\sum \int_{\mathbf{c}_i} \omega$, where ω is the differential form associated with the field $\mathbf{F}$, and $\mathbf{c}_i$ are the sides of the square with a fixed orientation, say, anticlockwise, so that

$$\begin{aligned}
\mathbf{c}_1 &= [(1,-1),(1,1)] \ , \\
\mathbf{c}_2 &= [(1,1),(-1,1)] \ , \\
\mathbf{c}_3 &= [(-1,1),(-1,-1)] \ , \\
\mathbf{c}_4 &= [(-1,-1),(1,-1)] \ ,
\end{aligned}$$

whence

$$\begin{aligned}
\int_{\mathbf{c}_1} \omega &= \int_{\mathbf{c}_1} x^2\,dy = \int_{-1}^{1} x^2\,dy = 2 \ , \\
\int_{\mathbf{c}_2} \omega &= \int_{\mathbf{c}_2} 2xy\,dx = -\int_{-1}^{1} 2xy\,dy = 0 \ , \\
\int_{\mathbf{c}_3} \omega &= \int_{\mathbf{c}_3} x^2\,dy = -\int_{-1}^{1} x^2\,dy = -2 \ , \\
\int_{\mathbf{c}_4} \omega &= \int_{\mathbf{c}_4} 2xy\,dy = \int_{-1}^{1} 2xy\,dy = 0 \ ,
\end{aligned}$$

and $I = 0$.

Alternatively, it follows from Stokes' formula, since the z-component of $\operatorname{curl} F$ is 0.

7.4.6 (4 points)

Find the area of the surface defined by $z = xy$ and $x^2 + y^2 \le 2$.

Solution: Since the surface is the graph of the roof function $h(x, y) = xy$ with partial derivatives $\frac{\partial h}{\partial x} = y$, $\frac{\partial h}{\partial y} = x$ over the domain $D = \{x^2 + y^2 \le 2\}$, its area is

$$\iint_D \sqrt{1 + x^2 + y^2}\, dx\, dy = \iint_D r\sqrt{1 + r^2}\, dr = 2\pi \int_0^{\sqrt{2}} r\sqrt{1 + r^2}\, dr$$
$$= \pi \int_0^{\sqrt{2}} \sqrt{1 + r^2}\, d(1 + r^2) = \pi \frac{2}{3} \left(1 + r^2\right)^{3/2} \Big|_{r=0}^{r=\sqrt{2}}$$
$$= \pi \frac{2}{3}(3\sqrt{3} - 1) .$$

7.4.10 (4 points)

Find the area of the portion S of the unit sphere which is cut out by the cone $z \ge \sqrt{x^2 + y^2}$.

Solution: By using spherical coordinates surface S can be parameterized as

$$x = \sin\phi \cos\theta ,$$
$$y = \sin\phi \sin\theta ,$$
$$z = \cos\phi$$

with $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$, so that

$$\text{area } S = \iint_S dA = \int_0^{2\pi} \int_0^{\pi/4} \|\mathbf{T}_\phi \times \mathbf{T}_\theta\|\, d\phi\, d\theta ,$$

where

$$\mathbf{T}_\phi = \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right) = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) ,$$
$$\mathbf{T}_\theta = \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) ,$$

whence

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = \left(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right) ,$$

and

$$\|\mathbf{T}_\phi \times \mathbf{T}_\theta\| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi} = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$
$$= \sin\phi .$$

Thus,

$$\text{area } S = 2\pi \int_0^{\pi/4} \sin\phi\, d\phi = -2\pi\cos\phi \Big|_{\phi=0}^{\phi=\pi/4} = \pi\left(2 - \sqrt{2}\right) .$$

7.5.20 (4 points)

Find the integral I of the function $f(x,y,z)=1-z$ over the graph of the function $h(x,y)=1-x^2-y^2$ with $x^2+y^2\le 1$.

Solution: Let D be the unit disk $\{x^2+y^2\le 1\}$, then

$$\begin{aligned}
I &= \iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}\,dx\,dy\\
&= \iint_D \left(x^2+y^2\right)\sqrt{1+4x^2+4y^2}\,dx\,dy\\
&= \int_0^{2\pi}\int_0^1 r^3\sqrt{1+4r^2}\,dr\,d\phi = \frac{\pi}{16}\int_0^1 \left(1+4r^2-1\right)\sqrt{1+4r^2}\,d\left(1+4r^2\right)\\
&= \frac{\pi}{16}\int_1^5 \left(u^{3/2}-u^{1/2}\right)du = \frac{\pi}{16}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_{u=1}^{u=5}\\
&= \frac{\pi}{16}\left(\frac{2}{5}5^{5/2}-\frac{2}{3}5^{3/2}-\frac{2}{5}+\frac{2}{3}\right) = \frac{\pi}{12}\left(5\sqrt{5}+\frac{1}{5}\right).
\end{aligned}$$

7.6.14 (4 points)

Find the surface integral $I=\iint_S \mathbf{F}\cdot\mathbf{n}\,dA$, where $\mathbf{F}(x,y,z)=\left(1,1,z\left(x^2+y^2\right)^2\right)$, and S is the surface of the cylinder $x^2+y^2\le 1$, $0\le z\le 1$.

Solution: The surface S is the disjoint union of surfaces

$$\begin{aligned}
S_1 &= \{x^2+y^2\le 1,\ z=0\}\ ,\\
S_2 &= \{x^2+y^2=1,\ 0\le z\le 1\}\ ,\\
S_3 &= \{x^2+y^2\le 1,\ z=1\}\ ,
\end{aligned}$$

so that $I=I_1+I_2+I_3$ is the sum of the integrals $I_i=\iint_{S_i}\mathbf{F}\cdot\mathbf{n}\,dA$ over the surfaces S_i, respectively. The outer normal fields for these surfaces are, respectively,

$$\mathbf{n}_1=(0,0,-1)\ ,\qquad \mathbf{n}_2(x,y,z)=(x,y,0)\ ,\qquad \mathbf{n}_3=(0,0,1)\ ,$$

so that

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{n}_1 &= -z\left(x^2+y^2\right)^2 = 0\ ,\\
\mathbf{F}\cdot\mathbf{n}_2 &= x+y\ ,\\
\mathbf{F}\cdot\mathbf{n}_3 &= z\left(x^2+y^2\right)^2 = \left(x^2+y^2\right)^2\ ,
\end{aligned}$$

whence $I_1=0$. The integral I_2 also vanishes from symmetry consideration (or one can show it directly by using cylindrical coordinates parametrization of the surface S_2). Finally, I_3 coincides with the integral of the function $f(x,y)=\left(x^2+y^2\right)^2$ over the unit disk D, i.e., with

$$\iint_D \left(x^2+y^2\right)^2 dx\,dy = \int_0^{2\pi}\int_0^r r^5\,dr\,d\phi = \frac{\pi}{3}\ ,$$

whence $I=\frac{\pi}{3}$.

7.6.22a (3 points)

Find $I = \iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the upper hemisphere $S = \{x^2 + y^2 + z^2 = 1,\ z \geq 0\}$ oriented by the normal pointing out of the sphere, and $\mathbf{F} = (x, y, 0)$.

Solution: The outer normal to S is $\mathbf{n}(x,y,z) = (x,y,z)$, whence $\mathbf{F} \cdot \mathbf{n} = x^2 + y^2$, so that

$$I = \iint_S \left(x^2 + y^2\right)\, dA\ .$$

Since S is the graph of the function $h(x,y) = \sqrt{1 - x^2 - y^2}$ over the unit disk $D = \{x^2 + y^2 \leq 1\}$, we have

$$\begin{aligned}
I &= \iint_D \left(x^2 + y^2\right) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\, dx\, dy \\
&= \iint_D \left(x^2 + y^2\right) \sqrt{1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2}}\, dx\, dy = \iint_D \frac{x^2 + y^2}{\sqrt{1 - x^2 - y^2}}\, dx\, dy \\
&= \int_0^{2\pi} \int_0^1 \frac{r^3}{\sqrt{1 - r^2}}\, dr\, d\phi = -\pi \int_0^1 \frac{r^2}{\sqrt{1 - r^2}}\, d\left(1 - r^2\right) = \pi \int_0^1 \frac{1 - t}{\sqrt{t}}\, dt \\
&= \pi \left(2\sqrt{t} - \frac{2}{3} t^{3/2}\right)\Big|_{t=0}^{t=1} = \frac{4}{3}\pi\ .
\end{aligned}$$

Alternatively, by symmetry considerations,

$$\iint_S x^2\, dA = \iint_S y^2\, dA = \iint_S z^2\, dA = \frac{1}{3} \iint_S \left(x^2 + y^2 + z^2\right)\, dA = \frac{1}{3}\, \text{area}\ S = \frac{2}{3}\pi\ .$$

7.6.22b (3 points)

Find $I = \iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the upper hemisphere $S = \{x^2 + y^2 + z^2 = 1,\ z \geq 0\}$ oriented by the normal pointing out of the sphere, and $\mathbf{F} = (y, x, 0)$.

Solution: In this case $\mathbf{F} \cdot \mathbf{n} = 2xy$, so that

$$I = \iint_S 2xy\, dA\ ,$$

which is equal 0 by symmetry considerations. Alternatively, in polar coordinates the angular part of I is

$$\int_0^{2\pi} \sin\phi \cos\phi\, d\phi = 0\ ,$$

whence $I = 0$.

4.4.4 (3 points)

Find $\operatorname{div}\mathbf{F}$ for $\mathbf{F}(x,y,z) = \big(x^2, (x+y)^2, (x+y+z)^2\big)$.

Solution: The corresponding partial derivatives of the components of F are:

$$\frac{\partial F_1}{\partial x} = 2x\ ,$$
$$\frac{\partial F_2}{\partial y} = 2(x+y)\ ,$$
$$\frac{\partial F_3}{\partial z} = 2(x+y+z)\ ,$$

whence

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} = 6x + 4y + 2z\ .$$

4.4.16 (3 points)

Find $\operatorname{curl}\mathbf{F}$ for

$$\mathbf{F} = \left(\frac{yz}{x^2+y^2+z^2}, -\frac{xz}{x^2+y^2+z^2}, \frac{xy}{x^2+y^2+z^2}\right)\ .$$

Solution:

$$\frac{\partial F_1}{\partial y} = \frac{z\,(x^2+y^2+z^2) - yz\cdot 2y}{(x^2+y^2+z^2)^2} = \frac{z\,(x^2+y^2+z^2) - yz\cdot 2y}{(x^2+y^2+z^2)^2}$$
$$= \frac{z\,(x^2-y^2+z^2)}{(x^2+y^2+z^2)^2}\ .$$

In the same way,

$$\frac{\partial F_1}{\partial z} = \frac{y\,(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}\ ,$$
$$\frac{\partial F_2}{\partial x} = -\frac{z\,(-x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}\ ,$$
$$\frac{\partial F_2}{\partial z} = -\frac{x\,(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}\ ,$$
$$\frac{\partial F_3}{\partial x} = \frac{y\,(-x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}\ ,$$
$$\frac{\partial F_3}{\partial y} = \frac{x\,(x^2-y^2+z^2)}{(x^2+y^2+z^2)^2}\ .$$

Therefore,

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \left(\frac{2x^3}{\left(x^2+y^2+z^2\right)^2}, \frac{2y\left(x^2-z^2\right)}{\left(x^2+y^2+z^2\right)^2}, \frac{-2z^3}{\left(x^2+y^2+z^2\right)^2} \right) \ .$$