$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Liner Alebra Exam Notes

lecture 1

Any liner system will have one of the following properties:

- NO solution
- 1 solution (uniqu)
- ∞ many

$$\underset{\text{row}}{\underset{\uparrow}{m}} \times \underset{\text{coloumn}}{\underset{\uparrow}{n}}$$

2 x 3: $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

REF: ① Rows of zero are on the bottom
② Any row with a leading entry must have zeros under

RREF: ① All leading entries are 1
② The leading 1 is the only non zero in it's coloumn

Elementary row Operations: ① Interchange rows $R_1 \longleftrightarrow R_2$
② Multiply a row by a constant
③ Add multiples of one row to anther

lecture 2

NO Solution: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$ Contradiction

Many solutions: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Z is a free variable

One Solution: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 2 & 1 \end{array}\right]$

Homo System: Every constant term on the right side is 0.

lecture 3

Multiplication of Matrices: let A be an $m \times n$ matrix

let B be an $n \times p$ matrix

$$A_{m \times n} \times B_{n \times p}$$

must be the same

Example: $A = \begin{bmatrix} -1 & 3 \\ 4 & 0 \\ 6 & 7 \end{bmatrix}_{3 \times 2}$ $B = \begin{bmatrix} 2 & 3 & -4 \\ 0 & 5 & -2 \end{bmatrix}_{2 \times 3}$

same

$$A \times B = \begin{bmatrix} -1 & 3 \\ 4 & 0 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 & -4 \\ 0 & 5 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (-1)(2)+(3)(0) & (-1)(3)+(3)(5) & (-1)(-4)+(3)(-2) \\ (4)(2)+(0)(0) & (4)(3)+(0)(5) & (4)(-4)+(0)(-2) \\ (6)(2)+(7)(0) & (6)(3)+(7)(5) & (6)(-4)+(7)(-2) \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & 12 & -2 \\ 8 & 12 & -16 \\ 12 & 53 & -38 \end{bmatrix}$$

$$AB \neq BA$$

Transpose: $m \times n$ matrix becomes $n \times m$

Example: $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -5 & -3 \end{bmatrix}_{2 \times 3}$ Symmetric if $A = A^T$

$$A^T = \begin{bmatrix} -1 & 0 \\ 2 & -5 \\ 3 & -3 \end{bmatrix}_{3 \times 2}$$

properties:
1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = A^T B^T$
5. $(A^r)^T = (A^T)^r$, for $r > 0$

Inverse: $A^{-1} \neq \frac{1}{A}$

$$AA^{-1} = A^{-1}A = I$$

Inverse of a 2x2 matrix: $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse for when $n > 2$: $[A / I] \xrightarrow[\text{row ops}]{\text{elementary}} [I / A^{-1}]$

lecture 4

A is not invertiable if this happens: $\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$

$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

properties: ① $(A^{-1})^{-1} = A$

② $(AB)^{-1} = B^{-1}A^{-1}$

③ $(kA)^{-1} = \frac{1}{k}A^{-1}$

④ $(A^{-1})^T = (A^T)^{-1}$

lecture 5

Elementary matrices: A matrix obtained by preforming a single elementary row operation on an identity matrix.

- Elementary matrices are invertable

Express A and A^{-1} as products as elementary matrices, to produce

$E_1, E_2, \ldots E_n$

where

$$E_1 E_2 \ldots E_n = A^{-1}$$
$$E_1^{-1} E_2^{-1} \ldots E_n^{-1} = A$$

lecture 6

$A^T A x = A^T y$

If $A^T A$ is invertible, then $x = (A^T A)^{-1} A^T y$

Example: Find the line of best fit for the given data

(1,2), (2,5), (3,3), (4,6)

$$y = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 6 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \quad A^T y = \begin{bmatrix} 16 \\ 45 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 16 \\ 45 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 45 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$b_0 = \frac{3}{2}$, $b_1 = 1$ $\quad y = x + \frac{3}{2}$

Derterminants: Derterminante of A can be denoted by det(A) or $|A|$

if $n = 2$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det A = a_{11} a_{22} - a_{21} a_{12}$$

if $n > 2$, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

Determinant of a 3x3 matrix:

$$\det A = \left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right|$$

$$(-)(-)(-) \quad (+)(+)(+) = \det A$$

- A square matrix is said to be an upper triangular matrix if the entries below the main diagonal are zero and lower triangular matrix if the entries above the main diagonal are zero.
  Then $\det A$ will be the values on the main diagonal multiplied.

lecture 7

Properties of determinants: let A and B be two $n \times n$ matrices

1. $\det A = \det A^T$
2. $\det AB = \det A \cdot \det B$
3. $kA = k^n \det A$, where k is a scalar
4. If A is invertible (i.e. A^{-1} exists) then $\det A^{-1} = \frac{1}{\det A}$
5. $\det A^r = (\det A)^r$, for any integer

- A square matrix is invertible iff $\det A \neq 0$

1. $\det B = -\det A$, if B is obtained from A by interchanging two different rows.
2. $\det B = k \det A$, if B is obtained from A by multiplying a row by a scalar.
3. $\det B = \det A$, if B is obtained by adding a multiple of a row to another.

$$\det(A+B) \neq \det A + \det B$$

let A be a square matrix
① If A has a row(column) of zeros, then det $A=0$
② If A has identical row(columns), then det $A=0$
③ If A has row(or columns) that are proportional, then det $A=0$

Cramers Rule: $x_1 = \dfrac{\det A_i(b)}{\det A}$

where $A_i(b)$ is obtained from A by replacing column i by b.

lecture 8 **Adjoint of a Matrix:** let A be an $n \times n$ matrix, such that $\det \neq 0$

$$A^{-1} = \frac{\text{adj } A}{\det A}$$

$M_{ij} = \det A_{ij}$

(i,j) cofactor

$C_{ij} = (-1)^{i+j} M_{ij}$

like determinants when $n > 2$, but do not include the number at ij. Also must do this for every number not just one row.

Make a matrix of the cofactors call B, the
adj $A = B^T$
Now $A^{-1} = \dfrac{\text{adj } A}{\det A}$

lecture 8 **Length (norm) of a vector in $\mathbb{R}^2$ and $\mathbb{R}^3$:**

$\vec{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the length of $\vec{u}$ is $\|\vec{u}\| = \sqrt{x_1^2 + x_2^2}$

• unit vector is a vector with length 1

If $\vec{v}$ is a vector in $\mathbb{R}^2$, then the unit vector in the direction of $\vec{v}$ is defined as $\vec{u} = \dfrac{1}{\|\vec{v}\|}\vec{v}$

Vector space and sub spaces: let V be a set of vectors, together with the two operations of addition and scalar multiplication.

V is a vector space if it has the following conditions

* ① If $\vec{u}$ and $\vec{v}$ are in V $\vec{u}+\vec{v}$ is in V
② $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
③ $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
④ $\vec{0}$ is in V, such that $\vec{0}+\vec{v}=\vec{v}$
⑤ For each $\vec{u}$ in $\vec{V}$, there is $\vec{u}$ such that $\vec{u}+(-\vec{u})=0$
* ⑥ If c is any s scalar, then $c\vec{v}\in V$
⑦ $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
⑧ $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
⑨ $c(d\vec{u})=(cd)\vec{u}$
⑩ $1\vec{u}=\vec{u}$

let S be a set $\mathbb{R}^n$, S is a subspace of $\mathbb{R}^n$ if the following is true:
① If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S *①
② If c is a scalar, then $c\vec{u}$ is in S *⑥

lecture 10 Spaning sets and linear Independence:
To check if vectors are a linear combination of a matrix, make an augmented matrix a solve.
If there is no solution, this is the unconsistant case, and there is no liner combination

To check if vectors span $\mathbb{R}^n$, choose any matrix in $\mathbb{R}^n$ and see if those vectors are a liner of the choosen vector.

linerally independent if $c_1=c_2=\dots=c_n=0$

lecture 11 **Basis and Dimension:** A basis for a vector space V, is a set of vectors of V such that

1. S is linearly independent
2. S spans V

linearly independent det $\neq 0$ (1)
to check if S spans V choose any vector in $\mathbb{R}^n$ and see if S can produce that vector.

- The dimension of a non-zero vector space V, it the number of vectors in a basis of V, written dim V.
  dim $\mathbb{R}^2 = 2$, dim $\mathbb{R}^3 = 3$

If you have V to be a vector space of dim n, and let S be a subset of V with exactly n vectors. Then S is a basis for V if either S spans V or S is linearly independent

Rank of a matrix: Turn your matrix into REF or RREF
Row space: non zero rows from the reduced form, taken from the reduced form
Col space: colounns with leading entries from reduced form, taken from the original

The rank of a matrix is equal to the number of non-zero rows in the reduced matrix.

lecture 12 $\dim(\text{col } A) = \dim(\text{row } A) = \text{rank}(A)$

rank of an $n \times n$ matrix A is $\text{Rank}(A) = \text{Rank}(A^T)$

Null Space: The solution space of the homo system $A\vec{x} = \vec{0}$, which is a subspace of $\mathbb{R}^n$. Dimension of null A is known as nullity of A.

If A is an $m \times n$ matrix, then $n = \text{rank}(A) + \dim(\text{null } A)$

Linear Transformations: $T: V \to W$, where V and W are vector spaces, is called a linear transformation if the following is satisfied:

① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}$ and $\vec{v}$ in V

② $T(c\vec{u}) = cT(\vec{u})$, for $\vec{u}$ in V any scalar c

lecture 13 Just more transformation examples

lecture 14 Eigen Values and Eigen vectors:

Let A be a matrix and $\vec{x}$ be an eigen vector

$$\underbrace{A\vec{x} = \lambda\vec{x}}_{\lambda\text{: eigen value}}$$, if not $\vec{x}$ is not an eigen vector

Eigen Values: calculate with the characteristic equation

$$\det(A - \lambda I) = 0$$

Eigen vectors: $(A - \lambda_i I)\vec{x} = 0$

lecture 15 If A is a triangular matrix then the eigen values of A, are the entries on it's main diagonal.

let A be an $n \times n$ matrix with distinct eigenvalues with corresponding eigen vectors. Then those eigen vectors will be linearly independent.

Diagonalization: If A and B are $n \times n$ matrices, then A is similar to B, $A \sim B$, if there is an invertible matrix P such that $B = P^{-1}AP \Leftrightarrow PB = AP$

• we say that a matrix A is diagonilizable if it is similar to a diagonal matrix. $P^{-1}AP = D \Leftrightarrow AP = PD$

2×2 matrix with eigen values λ_1, λ_2 and eigen vectors $\vec{v}_1, \vec{v}_2$.

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

lecture 16 $AP = PD \rightarrow$ Diagonal matrix

$\downarrow$ invertible matrix

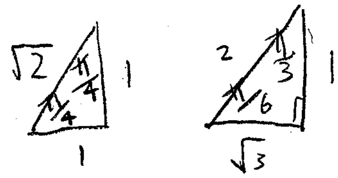
A is diagonalizable if $AP = PD$ is true.

Results: ① If A is an $n \times n$ matrix, and A has n real distinct roots, A is diagnolizable.

② If A is an $n \times n$ matrix and has n linearly independent eigen vectors, A is diagnolizable.

In general if k is a positive integer then $A^k = PD^kP^{-1}$

$$D^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$



lecture 17 **Complex Numbers:** A number in the form $z = a + ib$

- a and b are real numbers
- $i = \sqrt{-1}$, known as the imaginary unit, $i^2 = -1$

a is called the real part of z, written $Re(z) = a$

b is called the imaginary part of z, written $Im(z) = b$

Conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$, $z\bar{z}$ = real number

$z_1 \cdot z_2 = z_2 \cdot z_1$ $\qquad$ $(\bar{z})$ = real number

$|z| = \sqrt{a^2 + b^2}$, $|z|^2 = z\bar{z}$

$a = r\cos\theta$, $r = \sqrt{a^2 + b^2}$ $\qquad$ $z = a + ib$

$b = r\sin\theta$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + \sin\theta)$ ← polar form of z

lecture 18 **Complex eigen values and eigen vectors:**

If $\vec{v}_1$ is the eigen vector corresponding to λ_1, then $V_2 = \bar{\vec{V}}_1$, is the eigen vector corresponding to λ_2.

De Moiver's Theorm: For any positive integer n

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

lecture 19 when finding the eigen values for λ^3.

- By trial and error find 1 factor x
- Then take $\lambda - x$, and divide it into the equation to find the others

Inner product Spaces (dot product): denoted by $\langle \vec{u}, \vec{v} \rangle$ or $\vec{u} \cdot \vec{v}$

Example: $\vec{u} = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 5 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

$\vec{u} \cdot \vec{v} = (-2)(1) + (-3)(3) + (0)(2) + (5)(0)$

$= -2 - 9$

$= -11$

Results of Inner products: let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$ $(n > 0)$, and r be a real number

1. $\langle \vec{u}, \vec{v} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if $\vec{u} = \vec{0}$
2. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
3. $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u} + \vec{w} \rangle$
4. $r \langle \vec{u}, \vec{v} \rangle = \langle r\vec{u}, \vec{v} \rangle = \langle \vec{u}, r\vec{v} \rangle$

lecture 20 length or norm of $\vec{u}$ is $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

You can $\underbrace{\text{normalize}}_{\text{unit vector}}$ a vector by dividing it by it's length

Two $\underbrace{\text{vectors}}_{\text{(non zero)}}$ are said to be $\underbrace{\text{orthogonal}}_{\vec{u} \perp \vec{v}}$ if $\vec{u} \cdot \vec{v} = 0$ or perpendicular

If the length of each vector in S is equal to 1, then S is called an orthonormal set.

Orthogonal: In $\mathbb{R}^3$, $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$

orthonormal: In $\mathbb{R}^3$, $\|\vec{v}_1\| = 1$, $\|\vec{v}_2\| = 1$, $\|\vec{v}_3\| = 1$

- If S is an orthogonal set in $\mathbb{R}^n$, then S is a linearly independant set in $\mathbb{R}^n$
- If S is an orthogonal set in $\mathbb{R}^n$ set of non-zero vectors in $\mathbb{R}^n$, so forms a basis known as an orthogonal basis

- If the inner product space $\mathbb{R}^n$ has an orthogonal basis, then it also has an orthonormal basis

when we have an orthogonal basis, we can express the vectors as a liner combination with

$$C_i = \frac{\vec{x} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}, \quad i = 1, 2, \ldots k \qquad \text{to find } x = C_1\vec{v}_1 + \ldots C_n\vec{v}_n$$

lecture 21 **Orthogonal Projection:** let w be a non zero vector. Then the projection of $\vec{v}$ in the direction of $\vec{w}$ is the vector

$$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

in the direction of → $\vec{w}$; ↑ projection

The angle between vectors: let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^2$. The angle between them is given by:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

orthonormal basis: $\mathbb{R}^3$, $\vec{V}_1 \cdot \vec{V}_3 = 0$, $\vec{V}_1 \cdot \vec{V}_2 = 0$, $\vec{V}_2 \cdot \vec{V}_3 = 0$
$\vec{V}_1 \cdot \vec{V}_1 = 1$, $\vec{V}_2 \cdot \vec{V}_2 = 1$, $\vec{V}_3 \cdot \vec{V}_3 = 1$

In general: ① $\vec{V}_i \cdot V_j = 0 \quad i \neq j$
② $\vec{V}_i \cdot \vec{V}_j = 1 \quad i = j$

If $B = \{V, V_2 \ldots V_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Then for any vector $\vec{w}$ in $\mathbb{R}^n$ we have

$$\vec{w} = (\vec{w} \cdot \vec{V}_1)\vec{V}_1 + (\vec{w} \cdot \vec{V}_2)\vec{V}_2 + \ldots + (W \cdot \vec{V}_n) V_n$$

To express $\vec{w}$ as a linear combination of vectors in a set you must show that it is an orthonormal basis.

① show $\vec{V}_i \cdot \vec{V}_j = 0 \quad i = j$

② express V_i as a norm $\vec{u}_i$

Then $W = (\vec{w} \cdot \vec{u}_1) + (\vec{w} \cdot \vec{u}_2)\vec{u}_2 + \ldots + (\vec{w} \cdot \vec{u}_n)\vec{u}_n$

lecture 22 <u>The Gram-Schmidt Process:</u> suppose $B = \{b_1, b_2\}$ be a basis for $\mathbb{R}^2$

let $\vec{w}_1 = \vec{v}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot w_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1$$

then $C = \{w_1, w_2\}$ is an orthonormal basis for $\mathbb{R}^2$

In general $\vec{w}_1 = \vec{v}_1$

$$\vdots$$

$$\vec{w}_n = \vec{v}_n - \frac{\vec{v}_n \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} - \frac{\vec{v}_n \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 \cdots - \frac{\vec{v}_n \cdot \vec{w}_{n-1}}{\vec{w}_{n-1} \cdot \vec{w}_{n-1}} \vec{w}_{n-1}$$

Then $C = \{\vec{w}_1, \vec{w}_2 \ldots \vec{w}_n\}$ is an orthogonal basis for $\mathbb{R}^n$.

let $\vec{u}_i = \dfrac{\vec{w}_i}{\|\vec{w}_i\|}$, $i = 1, 2, \ldots, n$

Then $D = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_3\}$ is an orthonormal basis for $\mathbb{R}^n$.