

MAT 1322 E W2015 Wednesday, Feb. 11th 8:30–9:50 Prof. González

MIDTERM TEST 1

Max = 20

Student Number: _____

- Time: 80 min.
- Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.
- There are six questions worth a total of 20 marks.

1. (2 points) Consider the integral $\int_0^9 \frac{2}{(x-1)^{4/3}} dx$.

(a) Is this an improper integral and if yes, why?

It is an improper integral since the function $f(x) = \frac{2}{(x-1)^{4/3}}$ is not defined at $x = 1$ (it has a vertical asymptote at $x = 1$).

(b) Does it converge or diverge? If it converges, give its value. If it diverges, explain why.

The integral converges if the integrals $\int_0^1 \frac{2}{(x-1)^{4/3}} dx$ and $\int_1^9 \frac{2}{(x-1)^{4/3}} dx$ converge and then

$$\int_0^9 \frac{2}{(x-1)^{4/3}} dx = \int_0^1 \frac{2}{(x-1)^{4/3}} dx + \int_1^9 \frac{2}{(x-1)^{4/3}} dx.$$

It is divergent if any of this two integrals is divergent.

The indefinite integral is $\int \frac{2}{(x-1)^{4/3}} dx = \frac{-6}{(x-1)^{1/3}} + C$.

$$\int_0^1 \frac{2}{(x-1)^{4/3}} dx = \lim_{t \rightarrow 1^-} \int_0^t \frac{2}{(x-1)^{4/3}} dx = \lim_{t \rightarrow 1^-} \left. \frac{-6}{(x-1)^{1/3}} \right|_0^t = \lim_{t \rightarrow 1^-} \left(\frac{-6}{(x-1)^{1/3}} - \frac{-6}{-1} \right) = \infty$$

and then the integral $\int_0^9 \frac{2}{(x-1)^{4/3}} dx$ is divergent. Alternatively,

$$\int_1^9 \frac{2}{(x-1)^{4/3}} dx = \lim_{t \rightarrow 1^+} \int_t^9 \frac{2}{(x-1)^{4/3}} dx = \lim_{t \rightarrow 1^+} \left. \frac{-6}{(x-1)^{1/3}} \right|_t^9 = \lim_{t \rightarrow 1^+} \left(\frac{-6}{2} - \frac{-6}{(x-1)^{1/3}} \right) = \infty$$

and then the integral $\int_0^9 \frac{2}{(x-1)^{4/3}} dx$ is divergent.

2. (3 points) Determine if the statements below are true or false. Write T or F in the box provided.

(a) To determine if $\int_0^1 \frac{\sqrt{3+x}}{4x+x^2} dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

(i) $0 \leq \frac{\sqrt{3+x}}{4x+x^2} \leq \frac{\sqrt{4}}{4x+x^2} = \frac{2}{4x+x^2} \leq \frac{2}{x^2}$ for all $0 < x \leq 1$. True or False? True

(ii) $\int_0^1 \frac{2}{x^2} dx = 2 \int_0^1 \frac{1}{x^2} dx = \infty$. True or False? True

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_0^1 \frac{\sqrt{3+x}}{4x+x^2} dx$ is divergent by the comparison test. True or False? False

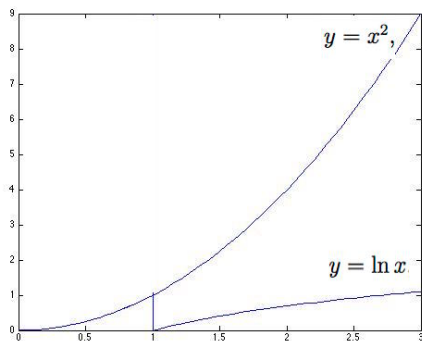
(b) To determine if $\int_1^\infty \frac{\sqrt{3+x}}{4x+x^2} dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

(i) $0 \leq \frac{\sqrt{3+x}}{4x+x^2} \leq \frac{\sqrt{4x}}{4x+x^2} \leq \frac{\sqrt{4x}}{5x^2} = \frac{2}{5x^{3/2}}$ for all $x \geq 1$. True or False? False

(ii) $\int_1^\infty \frac{2}{5x^{3/2}} dx = \frac{2}{5} \int_1^\infty \frac{1}{x^{3/2}} dx = \frac{4}{5}$ True or False? True

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_1^\infty \frac{\sqrt{3+x}}{4x+x^2} dx$ is convergent by the comparison test. True or False? True

3. (3 points) Sketch the region bounded by the curves $y = x^2$, $y = \ln x$, $x = 1$ and $x = 3$. What is the area of the region?



If $f(x) \geq g(x) \geq 0$ on the interval $[a, b]$, then the area between the curves is

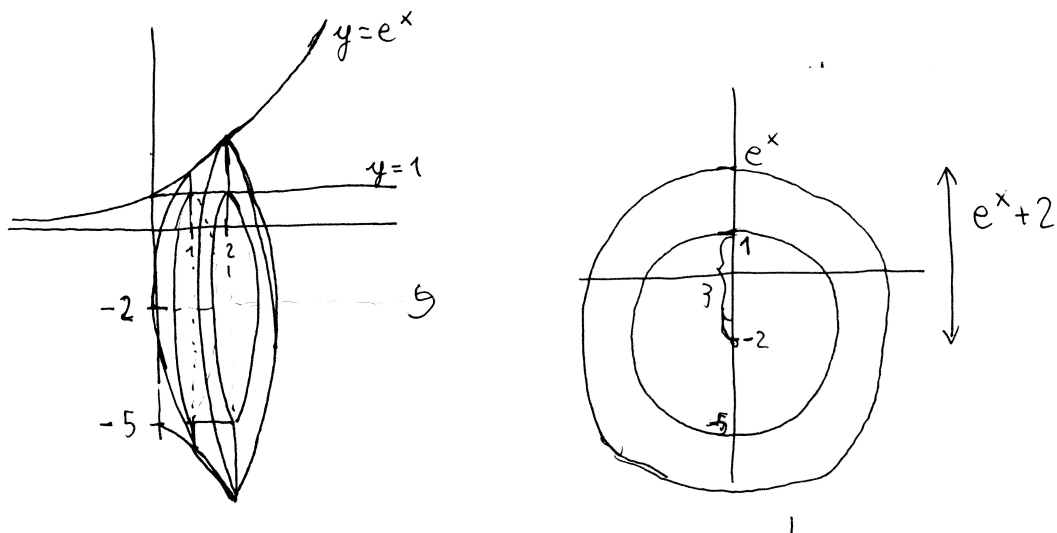
$$\begin{aligned} \text{Area} &= \int_a^b (f(x) - g(x)) dx \\ &= \int_1^3 (x^2 - \ln x) dx \end{aligned}$$

The integral $\int \ln x dx$ can be done by parts. Taking $u = \ln x$ and $dv = dx$ makes $du = dx/x$ and $v = x$. Then we have $\int \ln x dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + C$ which gives

$$\begin{aligned} \text{Area} &= \left(\frac{x^3}{3} - (x \ln x - x) \right) \Big|_1^3 \\ &= (9 - 3 \ln 3 + 3) - \left(\frac{1}{3} - 0 + 1 \right) = 32/3 - 3 \ln(3) \approx 7.3708 \end{aligned}$$

4. (4 points) Find the volume of the solid obtained when the region bounded by $y = e^x$, $y = 1$, $x = 1$ and $x = 2$ is rotated around the line $y = -2$.

(a) Sketch the region and a typical annular cross-section in a plane perpendicular to the x -axis.



(b) What are the inner radius r_{in} , the outer radius r_{out} and the area $A(x)$ of this ring?

Answers:

$$r_{in} = 3$$

$$r_{out} = e^x + 2$$

$$A(x) = \pi((e^x + 2)^2 - 3^2)$$

(c) Write the integral that gives the volume of the solid and evaluate it.

$$\begin{aligned} V &= \int_a^b A(x)dx = \int_1^2 \pi((e^x + 2)^2 - 3^2)dx \\ &= \pi \left(\int_1^2 (e^{2x} + 4e^x + 4)dx - 9 \int_1^2 dx \right) \\ &= \pi \left(\left. \frac{1}{2}e^{2x} + 4e^x + 4x \right|_1^2 - 9 \right) = \pi \left(\left(\frac{1}{2}e^4 + 4e^2 + 8 \right) - \left(\frac{1}{2}e^2 + 4e + 4 \right) - 9 \right) \\ &= \pi \left(\frac{1}{2}e^4 + \frac{7}{2}e^2 - 4e - 5 \right) \approx 37.2876\pi \approx 117.14 \end{aligned}$$

5. (4 points) A reservoir has the form of a cone as in the diagram.

Its height is 5 m, the radius at the top is 2 m, and it's full of water to a height of 4 m.

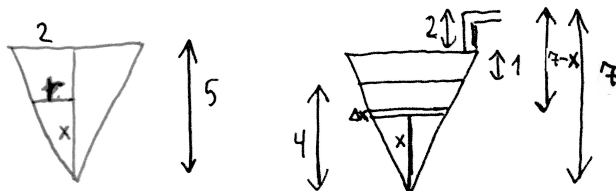
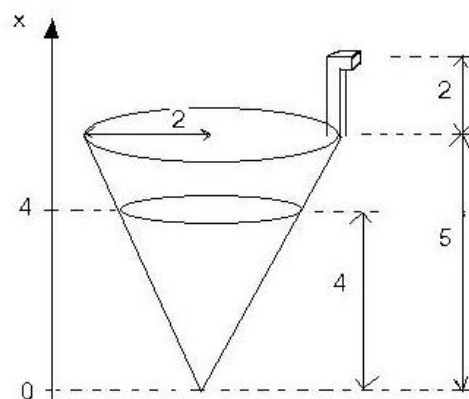
We want to pump all the water to a point 2 m above the reservoir.

Let x be the height measured in metres from the bottom of the reservoir.

(a) What is the radius r of the cone at height x ?

Answer: $r = x/2$

By similar triangles $r = 2x/5$



(b) Determine the approximate volume (in m^3) of a thin slice of the water between the heights x and $x + \Delta x$, for small Δx . Give your answer in the form $A(x)\Delta x$.

Answer: Volume = $\pi(2x/5)^2 \Delta x$

(c) Determine the approximate work (in J) required to pump this slice 2 m above the reservoir. Give your answer in the form $P(x)\Delta x$. Recall that the density of water is $\rho = 1000 \text{ kg/m}^3$ and that the acceleration of gravity is $g = 9.8 \text{ m/s}^2$.

Answer: Work = distance \times Force. And $F = g \cdot m = g \cdot \rho \cdot V$

$$\text{Work} = (3 + 4 - x)(m) \cdot 1000(\text{kg/m}^3) \cdot 9.8(\text{m/s}^2)(\pi(2x/5)^2 \Delta x)(\text{m}^3)$$

(d) Determine the work (in J) required to pump all the water in the reservoir to 2 m above it. Write the appropriate integral and evaluate it.

$$\begin{aligned} \text{Work} &= \int_0^4 (3 + 4 - x) \cdot 1000 \cdot 9.8 \pi (2x/5)^2 dx \\ &= 1568\pi \int_0^4 (7x^2 - x^3) dx \\ &= 1568\pi \left(7\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^4 = 1568\pi \left(7\frac{64}{3} - 64 \right) \approx 420353 \text{ J} \end{aligned}$$

6. (4 points)

(a) Give the integral for the average value of the function $f(x) = x \sin(x^2)$ on the interval $[0, \sqrt{\pi/2}]$ and then evaluate it.

The average value of a function $f(x)$ on an interval $[a, b]$ is given by

$$\begin{aligned} f_{ave} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{\sqrt{\pi/2} - 0} \int_0^{\sqrt{\pi/2}} x \sin(x^2) dx \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{2} - \cos(x^2) \Big|_0^{\sqrt{\pi/2}} = \frac{1}{\sqrt{2\pi}} (-\cos(\pi/2) + \cos(0)) = \frac{1}{\sqrt{2\pi}} \approx 0.3989 \end{aligned}$$

(b) Solve the initial value problem: $\frac{dy}{dx} = (1 + y^2) \sin x$, $y(0) = 0$.

This is a separable differential equation. Thus, in order to solve, we first separate the variables

$$\frac{dy}{(1+y^2)} = \sin x dx$$

And then we integrate

$$\begin{aligned} \int \frac{dy}{(1+y^2)} &= \int \sin x dx + C \\ \arctan y &= -\cos x + C \end{aligned} \tag{1}$$

Solving for y we get the general solution

$$y(x) = \tan(-\cos x + C)$$

Considering the initial condition $y(0) = 0$ in equation (1), we get

$$\arctan(0) = -\cos(0) + C$$

and thus $C = \arctan(0) + \cos(0) = 1$ and the particular solution is

$$y(x) = \tan(-\cos x + 1)$$

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1. (2 points) Consider the integral $\int_0^9 \frac{3}{(x-1)^{5/3}} dx$.

(a) Is this an improper integral and if yes, why?

It is an improper integral since the function $f(x) = \frac{3}{(x-1)^{5/3}}$ is not defined at $x = 1$ (it has a vertical asymptote at $x = 1$).

(b) Does it converge or diverge? If it converges, give its value. If it diverges, explain why.

The integral converges if the integrals $\int_0^1 \frac{3}{(x-1)^{5/3}} dx$ and $\int_1^9 \frac{3}{(x-1)^{5/3}} dx$ converge and then

$$\int_0^9 \frac{3}{(x-1)^{5/3}} dx = \int_0^1 \frac{3}{(x-1)^{5/3}} dx + \int_1^9 \frac{3}{(x-1)^{5/3}} dx.$$

It is divergent if any of this two integrals is divergent.

The indefinite integral is $\int \frac{3}{(x-1)^{5/3}} dx = \frac{-9}{2(x-1)^{2/3}} + C$.

We have

$$\int_0^1 \frac{3}{(x-1)^{5/3}} dx = \lim_{t \rightarrow 1^-} \int_0^t \frac{3}{(x-1)^{5/3}} dx = \lim_{t \rightarrow 1^-} \frac{-9}{2(x-1)^{2/3}} \Big|_0^t = \lim_{t \rightarrow 1^-} \left(\frac{-9}{2(t-1)^{2/3}} - \frac{-9}{2} \right) = \infty$$

and then the integral $\int_0^9 \frac{3}{(x-1)^{5/3}} dx$ is divergent. Alternatively,

$$\int_1^9 \frac{3}{(x-1)^{5/3}} dx = \lim_{t \rightarrow 1^+} \int_t^9 \frac{3}{(x-1)^{5/3}} dx = \lim_{t \rightarrow 1^+} \frac{-9}{2(x-1)^{2/3}} \Big|_t^9 = \lim_{t \rightarrow 1^+} \left(\frac{-9}{8} - \frac{-3}{2(t-1)^{1/3}} \right) = \infty$$

and then the integral $\int_0^9 \frac{3}{(x-1)^{4/3}} dx$ is divergent.

2. (3 points) Determine if the statements below are true or false. Write T or F in the box provided.

(a) To determine if $\int_0^1 \frac{\sqrt{8+x}}{x+3x^2} dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

(i) $\frac{\sqrt{8+x}}{x+3x^2} \geq \frac{\sqrt{8}}{x+3x^2} \geq \frac{\sqrt{8}}{x}$ for all $0 < x \leq 1$. True or False? *False*

(ii) $\int_0^1 \frac{\sqrt{8}}{x} dx = \sqrt{8} \int_0^1 \frac{1}{x} dx = \infty$. True or False? *True*

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_0^1 \frac{\sqrt{8+x}}{x+3x^2} dx$ is divergent by the comparison test. True or False? *True*

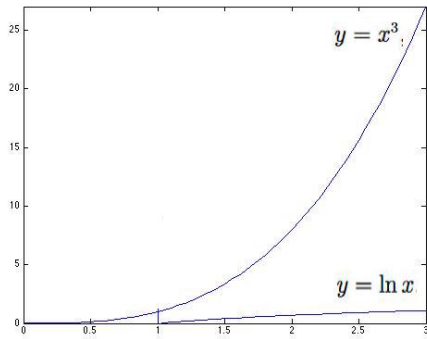
(b) To determine if $\int_1^\infty \frac{\sqrt{8+x}}{x+3x^2} dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

(i) $\frac{\sqrt{8+x}}{x+3x^2} \leq \frac{\sqrt{x}}{x+3x^2} \leq \frac{\sqrt{x}}{3x^2} = \frac{1}{3x^{3/2}}$ for all $x \geq 1$. True or False? *False*

(ii) $\int_1^\infty \frac{1}{3x^{3/2}} dx = \frac{1}{3} \int_1^\infty \frac{1}{x^{3/2}} dx = \frac{3}{4}$ True or False? *False*

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_1^\infty \frac{\sqrt{8+x}}{x+3x^2} dx$ is convergent by the comparison test. True or False? *True*

3. (3 points) Sketch the region bounded by the curves $y = x^3$, $y = \ln x$, $x = 1$ and $x = 3$. What is the area of the region?



If $f(x) \geq g(x) \geq 0$ on the interval $[a, b]$, then the area between the curves is

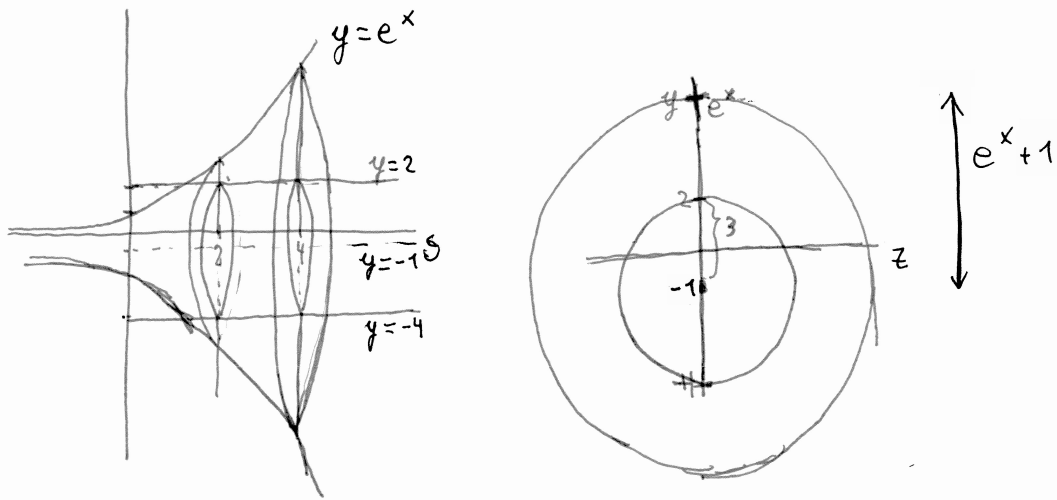
$$\begin{aligned} \text{Area} &= \int_a^b (f(x) - g(x)) dx \\ &= \int_1^3 (x^3 - \ln x) dx \end{aligned}$$

The integral $\int \ln x dx$ can be done by parts. Taking $u = \ln x$ and $dv = dx$ makes $du = dx/x$ and $v = x$. Then we have $\int \ln x dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + C$ which gives

$$\begin{aligned} \text{Area} &= \left(\frac{x^4}{4} - (x \ln x - x) \right) \Big|_1^3 \\ &= \left(\frac{81}{4} - (3 \ln 3 - 3) \right) - \left(\frac{1}{4} - (0 - 1) \right) \approx 18.7041 \end{aligned}$$

4. (4 points) Find the volume of the solid obtained when the region bounded by $y = e^x$, $y = 2$, $x = 2$ and $x = 4$ is rotated around the line $y = -1$.

(a) Sketch the region and a typical annular cross-section in a plane perpendicular to the x -axis.



(b) What are the inner radius r_{in} , the outer radius r_{out} and the area $A(x)$ of this ring?

Answers:

$$r_{in} = 3$$

$$r_{out} = e^x + 1$$

$$A(x) = \pi((e^x + 1)^2 - 3^2)$$

(c) Write the integral that gives the volume of the solid and evaluate it.

$$\begin{aligned} V &= \int_a^b A(x) dx = \int_2^4 \pi((e^x + 1)^2 - 3^2) dx \\ &= \pi \left(\int_2^4 (e^{2x} + 2e^x + 1) dx - 9 \int_2^4 dx \right) \\ &= \pi \left(\left. \frac{1}{2}e^{2x} + 2e^x + x \right|_2^4 - 18 \right) = \pi \left(\left(\frac{1}{2}e^8 + 2e^4 + 4 \right) - \left(\frac{1}{2}e^4 + 2e^2 + 2 \right) - 18 \right) \\ &= \pi \left(\frac{1}{2}e^8 + \frac{3}{2}e^4 - 2e^2 - 16 \right) \approx 1541.59\pi \approx 4843 \end{aligned}$$

5. (4 points) A reservoir has the form of a cone as in the diagram.

Its height is 4 m, the radius at the top is 2 m, and it's full of water to a height of 3 m.

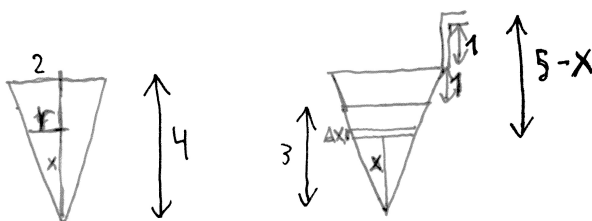
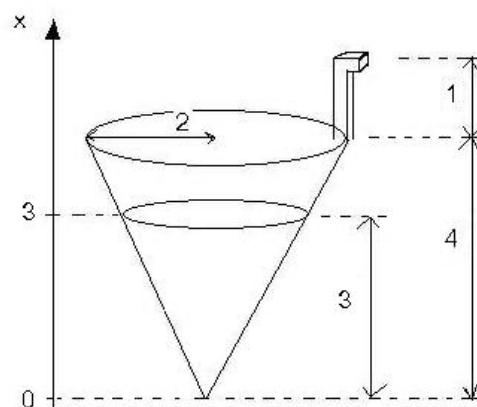
We want to pump all the water to a point 1 m above the reservoir.

Let x be the height measured in metres from the bottom of the reservoir.

(a) What is the radius r of the cone at height x ?

Answer: $r = x/2$

By similar triangles $r = x/2$



(b) Determine the approximate volume (in m^3) of a thin slice of the water between the heights x and $x + \Delta x$, for small Δx . Give your answer in the form $A(x)\Delta x$.

Answer: Volume = $\pi r^2 \Delta x = \pi(x/2)^2 \Delta x$

(c) Determine the approximate work (in J) required to pump this slice 1 m above the reservoir. Give your answer in the form $P(x)\Delta x$. Recall that the density of water is $\rho = 1000 \text{ kg/m}^3$ and that the acceleration of gravity is $g = 9.8 \text{ m/s}^2$.

Answer: Work = distance \times Force. And $F = g \cdot m = g \cdot \rho \cdot V$

Work = $(2 + 3 - x)(\text{m}) \cdot 1000(\text{kg/m}^3) \cdot 9.8(\text{m/s}^2)(\pi(x/2)^2 \Delta x)(\text{m}^3)$ (d) Determine the work (in J) required to pump all the water in the reservoir to 1 m above it. Write the appropriate integral and evaluate it.

$$\begin{aligned} \text{Work} &= \int_0^3 (2 + 3 - x) \cdot 1000 \cdot 9.8 \pi (x/2)^2 dx \\ &= 2450\pi \int_0^3 (5x^2 - x^3) dx \\ &= 2450\pi \left(5\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^3 = 2450\pi(45 - 81/4) \approx 190498 \text{ J} \end{aligned}$$

6. (4 points)

(a) Give the integral for the average value of the function $f(x) = x \cos(x^2)$ on the interval $[0, \sqrt{\pi/2}]$ and then evaluate it. The average value of a function $f(x)$ on an interval $[a, b]$ is given by

$$\begin{aligned} f_{ave} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{\sqrt{\pi/2} - 0} \int_0^{\sqrt{\pi/2}} x \cos(x^2) dx \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{2} \sin(x^2) \Big|_0^{\sqrt{\pi/2}} = \frac{1}{\sqrt{2\pi}} \sin(\pi/2) \approx 0.3989 \end{aligned}$$

(b) Solve the initial value problem: $\frac{dy}{dx} = \sqrt{1-y^2} \sin x$, $y(0) = 0$.

This is a separable differential equation. Thus, in order to solve, we first separate the variables

$$\frac{dy}{\sqrt{1-y^2}} = \sin x dx$$

And then we integrate

$$\begin{aligned} \int \frac{dy}{\sqrt{1-y^2}} &= \int \sin x dx + C \\ \arcsin y &= -\cos x + C \end{aligned} \tag{2}$$

Solving for y we get the general solution

$$y(x) = \sin(-\cos x + C)$$

Considering the initial condition $y(0) = 0$ in equation (2), we get

$$\arcsin(0) = -\cos(0) + C$$

and thus $C = \arcsin(0) + \cos(0) = +1$ and the particular solution is

$$y(x) = \sin(-\cos x + 1)$$