COMP 1805 – Discrete Structures
Assignment 3, Winter 2015

Due March 13 by **4:00pm** in the COMP 1805 drop box in HP 3115.

Print this assignment and answer all questions in the boxes provided. **Anything outside of the boxes will not be considered when marking your assignment. You must submit using this assignment sheet.** You must staple all of the pages of this assignment together (in the correct order) in the top-left corner of this page. Fill out your name and student number and circle your tutorial section in the space below, and fill out your student number and initials at the top of every page except this one.

Please note that HP 3115 is open Monday-Friday from 8:30am-4:30pm. It is your responsibility to submit while the room is open. **Absolutely no late assignments are accepted.**

All boxes will be marked out of 2: 2 points will be awarded for a correct answer, 1 point will be awarded for a partially correct answer (one major detail or a few minor details missing or wrong), and 0 points will be awarded for a completely incorrect or missing answer. You are always expected to justify your answer unless the question clearly indicates that it is not necessary to do so.

Name										TA Use Only	
Student #	S	O	L	U	T	I	O	N	S		
Tutorial	B1/M1			B2		B3		B5		Grade (/40)	TA

1. Compute closed forms (*i.e.*, equal expressions that do not contain a $\sum$) for the following summations.

(a) $\sum_{i=16}^{89} 4$

$= (89-16+1)\times 4 = 74\times 4 = 296$

(b) $\sum_{i=k}^{n} 2$

$= (n-k+1)\times 2$

(c) $\sum_{i=1}^{n}(2i+1)$

$$= \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 1$$

$$= 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= 2\,\frac{n(n+1)}{2} + n$$

(d) $\sum_{i=1}^{n}\sum_{j=i}^{m}(i+j)$

$$= \sum_{i=1}^{n}\left(\sum_{j=i}^{m} i + \sum_{j=i}^{m} j\right)$$

$$= \sum_{i=1}^{n}\left((m-i+1)i + \sum_{j=1}^{m} j - \sum_{j=1}^{i-1} j\right)$$

$$= \sum_{i=1}^{n}\left(im - i^2 + i + \frac{m(m+1)}{2} - \frac{i(i-1)}{2}\right)$$

$$= \sum_{i=1}^{n}\left(im - i^2 + i - \frac{1}{2}i^2 + \frac{1}{2}i + \frac{m(m+1)}{2}\right)$$

$$= \sum_{i=1}^{n}\left(im - \frac{3}{2}i^2 + \frac{3}{2}i + \frac{m(m+1)}{2}\right)$$

$$= m\sum_{i=1}^{n} i - \frac{3}{2}\sum_{i=1}^{n} i^2 + \frac{3}{2}\sum_{i=1}^{n} i + n\,\frac{m(m+1)}{2}$$

$$= m\,\frac{n(n+1)}{2} - \frac{3}{2}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{2}\,\frac{n(n+1)}{2} + n\,\frac{m(m+1)}{2}$$

(e) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{n}(k-1)$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\sum_{k=j}^{n} k - \sum_{k=j}^{n} 1\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\sum_{k=1}^{n} k - \sum_{k=1}^{j-1} k - (n-j+1)\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{n(n+1)}{2} - \frac{j(j-1)}{2} - n + j - 1\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{1}{2}n^2 + \frac{1}{2}n - \frac{1}{2}j^2 + \frac{1}{2}j - n + j - 1\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{1}{2}n^2 - \frac{1}{2}n - \frac{1}{2}j^2 + \frac{3}{2}j - 1\right)$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\left(\frac{1}{2}n^2 - \frac{1}{2}n - 1\right) - \frac{1}{2}\sum_{j=1}^{i} j^2 + \frac{3}{2}\sum_{j=1}^{i} j\right)$$

$$= \sum_{i=1}^{n}\left(i\left(\frac{1}{2}n^2 - \frac{1}{2}n - 1\right) - \frac{1}{2}\,\frac{i(i+1)(2i+1)}{6} + \frac{3}{2}\,\frac{i(i+1)}{2}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{1}{2}n^2 i - \frac{1}{2}ni - i - \frac{1}{6}i^3 - \frac{1}{4}i^2 - \frac{1}{12}i + \frac{3}{4}i^2 + \frac{3}{4}i\right)$$

$$= \sum_{i=1}^{n}\left(\frac{1}{2}n^2 i - \frac{1}{2}ni - \frac{1}{6}i^3 + \frac{1}{2}i^2 - \frac{1}{3}i\right)$$

$$= \frac{1}{2}n^2\sum_{i=1}^{n} i - \frac{1}{2}n\sum_{i=1}^{n} i - \frac{1}{6}\sum_{i=1}^{n} i^3 + \frac{1}{2}\sum_{i=1}^{n} i^2 - \frac{1}{3}\sum_{i=1}^{n} i$$

$$= \frac{1}{2}n^2\,\frac{n(n+1)}{2} - \frac{1}{2}n\,\frac{n(n+1)}{2} - \frac{1}{6}\,\frac{n^2(n+1)^2}{4} + \frac{1}{2}\,\frac{n(n+1)(2n+1)}{6} - \frac{1}{3}\,\frac{n(n+1)}{2}$$

2. How much time does an algorithm take to solve a problem of size n if this algorithm uses $2n^2 + 2^n$ operations, and each operation takes 10^{-9} seconds, given the following values of n? Note: if the answer is more than 60 seconds, answer in minutes; if the answer is more than 60 minutes, answer in hours; if the answer is more than 24 hours, answer in days; if the answer is more than 365 days, answer in years.

(a) $n = 10$

operations $= 2(10)^2 + 2^{10} = 1224$

time $=$ (# operations) $*$ (# operations per second)

$= 1224 * 10^{-9}$

$= 1.224 * 10^{-6}$ seconds

(b) $n = 20$

operations $= 2(20)^2 + 2^{20} = 1049376$

time $= (1049376) * 10^{-9}$

$= 1.049376 * 10^{-3}$ seconds

(c) $n = 50$

operations $= 2(50)^2 + 2^{50} = 1125899906847624$

time $= (1125899906847624) * 10^{-9}$

$\approx 1.1259 * 10^{6}$

≈ 13 days

(d) $n = 100$

operations $= 1267650600228229401496703225376$

time $=$ # operations $* 10^{-9}$

$\approx 1.267 * 10^{21}$ seconds

$\approx 4.017 * 10^{13}$ years [$\approx 2900 *$ age of universe!]

3. Let $f(n) = \sum_{i=1}^{n}\sum_{j=i}^{n} 3j$. Give a function $g(n)$ such that $f(n)$ is $O(g(n))$. Justify your answer.

$$f(n) = \sum_{i=1}^{n}\sum_{j=i}^{n} 3j$$

$$= 3\sum_{i=1}^{n}\sum_{j=i}^{n} j$$

$$= 3\sum_{i=1}^{n}\left(\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j\right)$$

$$= 3\sum_{i=1}^{n}\left(\frac{n(n+1)}{2} - \frac{i(i-1)}{2}\right)$$

$$= \frac{3}{2}\sum_{i=1}^{n}\left(n^2 + n - i^2 + i\right)$$

$$= \frac{3}{2}n^3 + \frac{3}{2}n^2 - \frac{3}{2}\sum_{i=1}^{n} i^2 + \frac{3}{2}\sum_{i=1}^{n} i$$

$$= \frac{3}{2}n^3 + \frac{3}{2}n^2 - \frac{3}{2}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{2}\,\frac{n(n+1)}{2}$$

$$= \frac{3}{2}n^3 + \frac{3}{2}n^2 - \frac{1}{2}n^3 - \frac{3}{4}n^2 - \frac{n}{4} + \frac{3}{4}n^2 + \frac{3}{4}n$$

$$= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$$

Let $g(n) = n^3$. We will show $f(n)$ is $O(g(n))$.

$$f(n) = n^3 + \frac{3}{2}n^2 + \frac{1}{2}$$

$$\leq n^3 + \frac{3}{2}n^3 + \frac{1}{2}n^3 \quad [\text{if } n \geq 1]$$

$$= 3n^3$$

$$= 3g(n)$$

$\therefore$ take $c = 3$, $k = 1$.

4. For the following definitions of $f(n)$ and $g(n)$, determine if $f(n)$ is $O(g(n))$, $f(n)$ is $\Omega(g(n))$, and/or $f(n)$ is $\Theta(g(n))$. Justify your answer.

(a) $f(n) = 3n^2 - 5n + 2$, $g(n) = n^2$

Big-Oh:

$f(n) = 3n^2 - 5n + 2$

$\leq 3n^2 + 2$ ($5n \geq 0 \Leftrightarrow n \geq 0$)

$\leq 3n^2 + 2n^2$ ($n^2 \geq 1$)

$= 5n^2$

$\therefore f(n) \leq 5g(n)$ for $n \geq 1$

$\therefore$ Yes, $f(n)$ is $O(g(n))$

Big-Omega:

$f(n) = 3n^2 - 5n + 2$

$\geq 3n^2 - 5n$

$= 2n^2 + (n^2 - 5n)$

$\geq 2n^2$ as long as $n^2 - 5n \geq 0$

$\Leftrightarrow n(n-5) \geq 0$

$\Leftrightarrow n \geq 5$

$\therefore f(n) \geq 2g(n)$ for $n \geq 5$

$\therefore$ Yes, $f(n)$ is $\Omega(g(n))$.

Since $f(n)$ is $O(g(n))$ and $\Omega(g(n))$, we have $f(n)$ is $\Theta(g(n))$.

(b) $f(n) = 5n^4 + 2n - 1$, $g(n) = n^3$

Big-Oh:

$f(n)$ is not $O(g(n))$.

Proof: suppose it is.

Then $5n^4 + 2n - 1 \leq cn^3$ for $n \geq k$.

Divide by n^3:

$\underbrace{5n + 2n^{-2} - n^{-3}}_{\text{line}} \leq \underbrace{c}_{\text{constant}}$

This is a contradiction!

$\therefore$ No, $f(n)$ is not $O(g(n))$.

Big-Omega:

$f(n) = 5n^4 + 2n - 1$

$\geq 5n^4 - 1$

$= 4n^4 + (n^4 - 1)$

$\geq 4n^4$ as long as $n^4 - 1 \geq 0$ ($\Leftrightarrow n \geq 1$.)

$\geq 4n^3$

$\therefore f(n) \geq 4g(n)$ for $n \geq 1$

$\therefore$ Yes, $f(n)$ is $\Omega(g(n))$

Since $f(n)$ is not $O(g(n))$, we have $f(n)$ is not $\Theta(g(n))$.

(c) $f(n) = 2n^4 - 5n^3 + 8n - 12$, $g(n) = n^5$

Big-Oh:

$f(n) = 2n^4 - 5n^3 + 8n - 12$

$\leq 2n^4 + 8n$ if $5n^3 \geq 0$

$\leq 2n^4 + 8n^4$ if $n^3 \geq 1$

$= 10n^4$

$\leq 10n^5$ if $n \geq 1$

$\therefore f(n) \leq 10\, g(n)$ for $n \geq 1$

$\therefore$ Yes, $f(n)$ is $O(g(n))$.

Big-Omega:

$f(n)$ is not $\Omega(g(n))$.

Proof: suppose it is.

Then $2n^4 - 5n^3 + 8n - 12 \geq cn^5$ for $n \geq k$.

Divide by n^5:

$\underbrace{2n^{-1} - 5n^{-2} + 8n^{-4} - 12n^{-5}}_{\text{decreases} \to 0 \text{ as } n \text{ increases}} \geq c$ ← constant

This is a contradiction!

$\therefore$ No, $f(n)$ is not $\Omega(g(n))$

Since $f(n)$ is not $\Omega(g(n))$, we have that $f(n)$ is not $\Theta(g(n))$.

(d) $f(n) = n \log n$, $g(n) = n$

Big-Oh:

$f(n)$ is not $O(g(n))$.

Proof: suppose it is.

Then $n \log n \leq cn$ for $n \geq k$.

Divide by n:

$\underbrace{\log n}_{\text{increases to infinity}} \leq \underbrace{c}_{\text{constant}}$

This is a contradiction!

$\therefore$ No, $f(n)$ is not $O(g(n))$.

Big-Omega:

$f(n) = n \log n$

$\geq n$ as long as $\log n \geq 1$

$\iff n \geq$ base of log, say b.

$\therefore f(n) \geq n$ for $n \geq b$

$\therefore$ Yes, $f(n)$ is $\Omega(g(n))$.

Since $f(n)$ is not $O(g(n))$, we have that $f(n)$ is not $\Theta(g(n))$.

(e) $f(n) = 2^n$, $g(n) = 3^n$

Big-Oh

$f(n) = 2^n \le 3^n$

$\therefore f(n) \le g(n)$ for $n \geq 1$.

$\therefore$ Yes, $f(n)$ is $O(g(n))$.

Since $f(n)$ is not $\Omega(g(n))$, we have that $f(n)$ is not $\Theta(g(n))$.

Big-Omega:

$f(n)$ is not $\Omega(g(n))$

Proof: Suppose it is.

Then $2^n \geq c3^n$ for $n \geq k$.

Divide by 3^n:

$$\frac{2^n}{3^n} \geq c$$

$$\underbrace{\left(\tfrac{2}{3}\right)^n}_{\text{decreases} \to 0} \geq \underbrace{c}_{\text{constant}}$$

This is a contradiction

$\therefore$ No, $f(n)$ is not $\Omega(g(n))$.

5. Suppose that $f(n)$ is $O(g(n))$. Prove or disprove that $2^{f(n)}$ is $O(2^{g(n)})$.

This is false.

Let $f(n) = n^2 + n$ and $g(n) = n^2$.

Then $f(n)$ is $O(g(n))$ because $n^2 + n \le n^2 + n^2 = 2n^2$ (take $c = 2$, $k = 1$).

However, 2^{n^2+n} is not $O(2^{n^2})$.

Proof: Assume it is. Then $2^{n^2+n} \le c\,2^{n^2}$ for $n \geq k$.

Divide both sides by 2^{n^2}:

$$\frac{2^{n^2+n}}{2^{n^2}} \le c \iff 2^{n^2+n-n^2} \le c \iff \underbrace{2^n}_{\text{increases} \to \infty} \le \underbrace{c}_{\text{constant}}$$

This is a contradiction! $\therefore 2^{f(n)}$ is not $O(2^{g(n)})$.

6. Suppose that $f(n)$ is $\Theta(g(n))$ and $g(n)$ is $\Theta(h(n))$. Prove or disprove that $f(n)$ is $\Theta(h(n))$.

This is true.

$f(n)$ is $\theta(g(n))$ $<$ $f(n)$ is $O(g(n)) \Leftrightarrow f(n) \le c_1 g(n)$ for $n \ge k_1$
$f(n)$ is $\Omega(g(n)) \Leftrightarrow f(n) \ge c_2 g(n)$ for $n \ge k_2$

$g(n)$ is $\theta(h(n))$ $<$ $g(n)$ is $O(h(n)) \Leftrightarrow g(n) \le c_3 h(n)$ for $n \ge k_3$
$g(n)$ is $\Omega(h(n)) \Leftrightarrow g(n) \ge c_4 h(n)$ for $n \ge k_4$

Now:

$f(n) \le c_1 g(n) \le c_1(c_3 h(n))$ for $n \ge \max(k_1, k_3)$

$\therefore f(n)$ is $O(h(n))$ with $c = c_1 c_3$, $k = \max(k_1, k_3)$

$f(n) \ge c_2 g(n) \ge c_2(c_4 h(n))$ for $n \ge \max(k_2, k_4)$

$\therefore f(n)$ is $\Omega(h(n))$ with $c = c_2 c_4$, $k = \max(k_2, k_4)$.

Since $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(h(n))$, we have $f(n)$ is $\Theta(h(n))$.

7. Prove or disprove that n^n is $O(n!)$.

This is false.

Proof: Assume that n^n is $O(n!)$.

Then $n^n \le cn!$ for $n \ge k$.

Divide by $n!$ to get $\frac{n^n}{n!} \le c$

$$\frac{n^n}{n!} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \frac{n}{n-3} \cdot \ldots \cdot \frac{n}{2} \cdot \frac{n}{1}$$

Each factor is ≥ 1, and last is $= n$.

Therefore, $\frac{n^n}{n!} \ge n$, which means it cannot be $\le c$. This is a contradiction.

$\therefore n^n$ is <u>not</u> $O(n!)$

8. Prove that $2^n > n^2 + n$ for all integers $n > 4$.

Proof by induction.

Basis step: $(n=5)$ $LHS = 2^5 = 32$

$RHS = 5^2 + 5 = 25 + 5 = 30$

$LHS > RHS$, $\therefore$ true.

Inductive Hypothesis: Assume $2^k > k^2 + k$ for some integer k.

Inductive Step: Must show $2^{k+1} > (k+1)^2 + (k+1)$.

Start on right side:

$$\begin{aligned}(k+1)^2 + (k+1) &= (k^2 + 2k + 1) + (k+1) \\ &= k^2 + 3k + 2 \\ &< k^2 + 4k && \text{since } k > 2 \\ &< k^2 + k^2 && \text{since } 4k < k^2 \Leftrightarrow k > 4 \\ &= 2k^2 \\ &< 2(k^2 + k) && \text{since } k^2 + k > k \\ &< 2 \cdot 2^k && \text{by IH} \\ &= 2^{k+1}\end{aligned}$$

$\therefore (k+1)^2 + (k+1) < 2^{k+1}$, so inductive step is true.

9. Prove that $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ for all positive integers n.

Proof by induction.

Basis step: $(n=1)$ $LHS = \sum_{i=1}^{1} i2^i = 1 \cdot 2^1 = 2$

$RHS = (1-1)2^{1+1} + 2 = 0 + 2 = 2$

$LHS = RHS$, $\therefore$ true.

Inductive Hypothesis: Assume $\sum_{i=1}^{k} i2^i = (k-1)2^{k+1} + 2$.

Inductive step: Want to show $\sum_{i=1}^{k+1} i2^i = k2^{k+2} + 2$.

start on left side:

$$\sum_{i=1}^{k+1} i2^i = \sum_{i=1}^{k} i2^i + (k+1)2^{k+1}$$

$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1} \quad \text{(by IH)}$$

$$= (k-1+k+1)2^{k+1} + 2$$

$$= (2k)2^{k+1} + 2$$

$$= k2^{k+2} + 2$$

$\therefore \sum_{i=1}^{k+1} i2^i = k2^{k+2} + 2$, so inductive step is true.