# Midterm I: Review

**Chapters 1-7:** Review of the derivative
- limits
- definiton of the derivative
- derivative of polynomials, exponentials, log, and trig functions
- derivative rules: product, quotient, and chain rule.

## Chapter 8

### Section 8.1: Antiderivatives and the rules of integration
- power rule
- integrating sums and differences
- constant multiple
- exponents, logs, trig

### Section 8.2: Integration by Substitution
- Solve integrals in the form

$$\int f(g(x))g'(x)dx$$

Let $u=g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$

Integral becomes

$$\int f(u)du$$

- Undoes Chain rule

### Section 8.3: Area and the definite integral

- Calculate area under curve using Riemann Sum

$$\lim_{n\to\infty}\left(\Delta x f(x_1) + \ldots + \Delta x f(x_n)\right)$$

- Definite Integral $\int_a^b f(x)dx$ = area under curve if $f(x) \geq 0$.

## Section 8.4: The Fundamental Theorem of Calculus

- Theorem: Let $f(x)$ be a continuous function on $[a,b]$. Let $F(x)$ be any antiderivative of $f(x)$, $(F'(x)=f(x))$, then $\int_a^b f(x)dx$ exists and $\int_a^b f(x)dx = F(b)-F(a)$.
- Proof
  - why does the antiderivative give us area?
  - Why does it only depend on endpoints

## Section 8.5: Evaluating definite integrals

- Properties of definite integrals:
  - $\int_a^a f(x)dx = 0$
  - $\int_a^b f(x)dx = -\int_b^a f(x)dx$
  - $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
  - $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
  - $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

- Evaluating definite integrals
- Average value of a function: $\frac{1}{b-a}\int_a^b f(x)dx$

## Section 8.6: Area between two curves

- Finding the area between two curves:
  - $\int_a^b f(x)-g(x)dx$ if $f(x) \geq g(x)$

## Section 9.1: Integration by Parts

- $\int u\,dv = uv - \int v\,du$

- undoes the product rule

## Section 9.5: Improper Integrals

- Definitions:
  - □ $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$
  - □ $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$
  - □ $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$
- Convergent if integral gives a finite value.
  Divergent if integral gives infinite value.
- Be careful! Some improper integrals are disguised, eg $\int_{-1}^{1} \frac{1}{x}dx$.

## Tips when integrating

- Definite integrals give you a number.
  - → For example, integration by parts gives

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- Indefinte integrals or antiderivatives are functions
  - → Remember to include the constant

$$\int f(x)\,dx = F(x) + C$$

- We learnt two techniques for simplifying integrals
  - → Substitution
  - → Integration by parts

- It is not necessarily obvious how to solve an integral. I won't be giving any extra clues on the exam.
  - → Look at the different components of the integrand

- Substitution
  - → Look for derivatives of different substitution choices.
  - → For example

$$\int \frac{1}{5} x^4 e^{x^5}\,dx \implies$$ $u = x^5$ is a good choice because $\frac{du}{dx} = \frac{1}{5}x^4$ is in the integrand

- Integration by parts
  - → $du$ is simpler than $u$
  - → $dv$ is easy to integrate
  - → A good example is

$$\int \underset{\uparrow \atop u}{x}\, \underbrace{f(x)\,dx}_{dv}$$

- Look at improper integrals

## Example

$$\int \frac{2x}{x^2+1}\,dx$$

what method should we use?

Let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln(u) + C$$

$$= \ln(x^2+1) + C$$

## Example

$$\int 2x \ln(5x)\,dx$$

what method should we use?

Let $u = \ln(5x)$ and $dv = 2x\,dx$

$\Rightarrow \frac{du}{dx} = \frac{5}{5x} = \frac{1}{x}$ $\qquad \Rightarrow v = x^2$

$\Rightarrow du = \frac{dx}{x}$

Therefore

$$\int 2x \ln(5x)\,dx = x^2 \ln(5x) - \int \overbrace{x^2 \frac{1}{x}}^{x}\,dx$$

$$= x^2 \ln(5x) - \frac{1}{2}x^2 + C$$

## Example

$$\int (x+2)e^{2x}\,dx$$ what method?

Let $u = x+2$ and $dv = e^{2x}dx$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow v = \frac{1}{2}e^{2x}$

$\Rightarrow du = dx$

Therefore,

$$\int (x+2)e^{2x}\,dx = \frac{1}{2}(x+2)e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \frac{1}{2}(x+2)e^{2x} - \frac{1}{4}e^{2x} + C$$

## Example

$$\int x^2 e^{1-x^3}\,dx$$ what method?

Let $u = 1-x^3 \Rightarrow \frac{du}{dx} = -3x^2 \Rightarrow du = -3x^2dx$

Therefore,

$$\int x^2 e^{1-x^3}\,dx = \int \frac{-1}{3}e^{u}\,du = \frac{-1}{3}e^{u} + C$$

$$= \frac{-1}{3}e^{1-x^3} + C$$

Example

$$\int e^{-\sqrt{x}}\,dx$$

What method?
Not clear
Try substitution

Let $s=\sqrt{x} \Rightarrow \frac{ds}{dx}=\frac{1}{2}\frac{1}{\sqrt{x}} \Rightarrow ds=\frac{dx}{2\sqrt{x}}=\frac{dx}{2s}$

Therefore,

$$\int e^{-\sqrt{x}}\,dx=\int 2se^{-s}\,ds$$

Now, what method?

Let $u=2s$ and $dv=e^{-s}ds$

$\Rightarrow \frac{du}{ds}=2$ $\qquad \Rightarrow v=-e^{-s}$

$\Rightarrow du=2ds$

Therefore

$$\int 2se^{-s}ds=-2se^{-s}+\int 2e^{-s}ds$$

$$=-2se^{-s}-2e^{-s}+C$$

$$=-2\sqrt{x}e^{-\sqrt{x}}-2e^{-\sqrt{x}}+C$$

## Example

$\int x^2 \sin(x)\,dx$ what method?

Let $u = x^2$ and $dv = \sin(x)\,dx$

$\Rightarrow \frac{du}{dx} = 2x$ $\Rightarrow v = -\cos(x)$

$\Rightarrow du = 2x\,dx$

Therefore,

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + \underline{\int 2x\cos(x)\,dx}$$

↪ what method?

Let $u = 2x$ and $dv = \cos(x)\,dx$

$\Rightarrow \frac{du}{dx} = 2$ $v = \sin(x)$

$\Rightarrow du = 2\,dx$

Therefore,

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx$$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$