MAT 1341A F14 Final

1. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x \geq 0,\ y \geq 0 \text{ and } z \geq 0\}$. Which one of the following statements is true?

A. $(0, 0, 0) \in X$ and X is closed under multiplication by scalars. F

B. $(0, 0, 0) \notin X$ but X is closed under addition. F $(0,0,0) \in X$

C. X is closed under addition but X is not closed under multiplication by scalars. T

D. X is closed under addition and X is closed under multiplication by scalars. F

E. X is not closed under addition but X is closed under multiplication by scalars. F

F. None of the other statements is true.

- X is not closed under mult^n by scalars: $v = (1,0,0) \in X$ but $-1 \cdot v = (-1,0,0) \notin X$.
- X is closed under add^n ("+ve + +ve is +ve")

2. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent if $a = b = c = 0$ implies $au + bv + cw = 0$. F

II. A set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ implies $a = b = c = 0$. T (def^{n})

III. A set $\{u, v, w\} \subset V$ of vectors spans a vector space V if every vector in V is a linear combination of u and w. T

IV. $\{(1, -1), (1, 1)\}$ is an orthogonal set in $\mathbf{R}^2$. T

A. Only I & II

B. Only II & IV

C. Only II & III

D. Only I & III & IV

E. Only II & III & IV

F. All of the above statements are true.

3. In a linear system $Ax = b$, with n equations and n unknowns, the rank of A is $n-1$ and the rank of the augmented matrix $[A \mid b]$ is also $n-1$. Which one of the following statements is true?

A. The system has no solution. F

B. The system has a unique solution. F rank $A = n-1 < n = \#$ variables

C. The system has infinitely many solutions. T rank $A =$ rank $A|b < \#$ variables

D. The system has exactly $n-1$ solutions.

E. The determinant of A is non-zero.

F. Such a system cannot exist.

4. If C is an $m \times 2$ matrix and $D = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$, then the second column of the matrix CD is

A. not defined unless $m = 2$.

B. twice the first column of C.

C. the same as the first column of C.

D. the same as the second column of C.

E. the sum of the first and the second columns of C.

F. the sum of twice the first column of C and three times the second column of C.

$C = [C_1 C_2]$ $CD = [C_1 C_2] \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

$= [C_2 \quad 2C_1 + 3C_2 \quad C_1 + C_2]$

5. The dimension of $S = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid A = A^t\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

$$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \,\middle|\, a, b, d \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}.$$

Moreover, $\{M_1, M_2, M_3\}$ is l.i., so dim $S = 3$

6. Let A be a 7×12 matrix such that $\operatorname{rank} A = 6$. As usual, $\ker A = \{x \in \mathbf{R}^{12} \mid Ax = 0\}$ and col $A = \{Ax \mid x \in \mathbf{R}^{12}\}$. Which of the following statements is true?

A. $\dim \operatorname{col} A = 6, \dim \ker A = 1$

B. $\operatorname{col} A = \mathbf{R}^{12}, \dim \ker A = 6$

C. $\operatorname{col} A = \mathbf{R}^{7}, \dim \ker A = 5$

D. $\dim \operatorname{col} A = 5, \ker A = \mathbf{R}^{12}$

E. $\dim \operatorname{col} A = 6, \dim \ker A = 6$

F. $\operatorname{col} A = \{0\}, \dim \ker A = 5$

$$\begin{array}{cc} & 12 \\ 7 & \boxed{A \quad 0} \end{array}$$

rank $A = 6$.

$\therefore$ dim ker $A = 12 - 6 = 6$.
dim col A = rank $A = 6$

7. If two $n \times n$ matrices A and B satisfy $A^t = B^{-1}$ and $B^t = -B^{-1}$ then $(ABA)^t$ is always

A. $-B^3$
B. B^2A
C. $-B^{-3}$
D. B^{-3}
E. B^3
F. AB^2

$(ABA)^t = A^t B^t A^t$
$= B^{-1}(-B^{-1})B^{-1}$
$= -B^{-3}$

8. Which two of the following statements are **false**?

(i) For all invertible $n \times n$ matrices A and B, $\det(A^{-1}BA) = \det B$ T

(ii) For all invertible $n \times n$ matrices A and B, $\det(A^{-1}B^{-1}AB) = 1$ T

(iii) For all $n \times n$ matrices A and B, $(A^tB^t)^t = AB$ F $(A^tB^t)^t = BA$

(iv) For all invertible $n \times n$ matrices A and B, $(ABA^{-1})^{-1} = A^{-1}B^{-1}A$ F $(ABA^{-1})^{-1} = AB^{-1}A^{-1}$

(v) For all $n \times n$ matrices A and B, $\det(A^tB) = \det(B^tA)$ T

A. (i) and (iii)

B. (ii) and (iii)

C. (iii) and (iv)

D. (ii) and (iv)

E. (ii) and (v)

F. (i) and (v)

9. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix}$.

$$= 4\begin{vmatrix} g & a & d-2a \\ h & b & e-2b \\ i & c & f-2c \end{vmatrix} \quad 2C_2 + C_3 \to C_3$$

$$= 4\begin{vmatrix} g & a & d \\ h & b & e \\ i & c & f \end{vmatrix}$$

$$= 4\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} \quad (\det A = \det A^t)$$

$$= -4\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

$$= +4\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 12$$

A. 24
B. -24
C. 12
D. -12
E. 6
F. -6

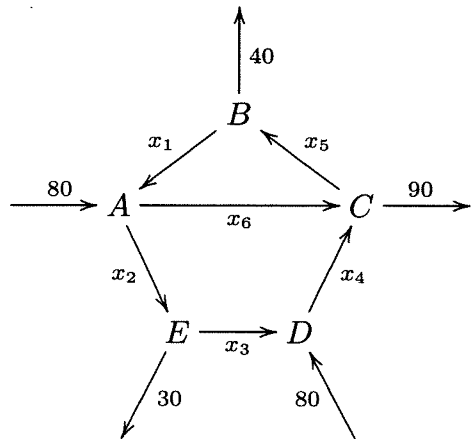
10. Which of the following are linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{\cos x, \sin x\}$
$T = \{1, \cos^2 x, \sin^2 x\}$
$U = \{1, 2\cos^2 x, 3\sin^2 x\}$
$V = \{1, \cos x, \sin x\}$

We know $\sin^2 x + \cos^2 x = 1$, $\forall x$, so T and U are dependent. Hence the answer must be D

A. T and V.
B. T and U.
C. S and T.
D. S and V.
E. S, U and V.
F. S, U and T.

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables x_i, $i = 1, \ldots, 6$.
(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection	Flow in	=	Flow out
A	$80 + x_1$	=	$x_2 + x_6$
B	x_5	=	$x_1 + 40$
C	$x_4 + x_6$	=	$x_5 + 90$
D	$80 + x_3$	=	x_4
2 E	x_2	=	$30 + x_3$

Constraints. $x_i \geq 0$, $i = 1, \ldots, 6$ (One-way streets)

1 $x_i \in \mathbb{Z}$ (# of cars.)

11(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\begin{array}{c} \begin{array}{cccccccc} & & & & s & t & & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

2

$$\begin{aligned} x_1 &= -40 + s \\ x_2 &= 40 + s - t \\ x_3 &= 10 + s - t \\ x_4 &= 90 + s - t \\ x_5 &= s \\ x_6 &= t \end{aligned} \quad ; \quad s, t \in \mathbb{R}$$

c) If $\overline{AC}$ were closed due to roadwork, find the minimum flow along $\overline{ED}$, **using your results from (b).**
(You must justify all your answers.)

If $\overline{AC}$ is closed, $x_6 = t = 0$. We look for the minimum of $\underline{x_3}$.
Note $x_3 = 10 + s$, so we seek the minimum of s, first.

Constraints.

$$\left.\begin{aligned} x_1 \geq 0 &\Leftrightarrow s \geq 40 \\ x_2 \geq 0 &\Leftrightarrow s \geq -40 \\ x_3 \geq 0 &\Leftrightarrow s \geq -10 \\ x_4 \geq 0 &\Leftrightarrow s \geq -90 \\ x_5 \geq 0 &\Leftrightarrow s \geq 0 \end{aligned}\right\} \Rightarrow \quad s \geq 40$$

$\therefore\ x_3 = 10 + s \geq \underline{50}.$

$\frac{1}{2}$

$\frac{1}{2}$ - justn.

$x + z + w = 0$

12. Let $v_1 = (1,0,0,-1)$, $v_2 = (1,-1,0,0)$, $v_3 = (1,0,1,0)$ and set $U = \text{span}\{v_1, v_2, v_3\} \subset \mathbf{R}^4$.

$\{(x,y,z,w) \mid x+y-z+w=0\}$

1.5 a) Briefly explain (you may refer to results learned in class or from the book) why U is a subspace of $\mathbf{R}^4$, and why $\{v_1, v_2, v_3\}$ is a basis for U.

$x+y+w=z$

2 b) Use the **Gram-Schmidt algorithm** to find an orthogonal basis for U.

2.5 c) Find the best approximation by a vector in U to the vector $(2,0,2,4)$.

a) U is a subspace because it is the span of v_1, v_2 & v_3. 1/2

$\{v_1, v_2, v_3\}$ is a basis for U since it spans U (by defn) and is l.i. 1/2 because $\text{rank}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 3.$ 1/2

So U (being the rowspace of ↑) has dimension 3.

b) We apply Gram Schmidt: 1 (correct)

Set $u_1 = v_1$

$\tilde{u}_2 = v_2 - \dfrac{v_2 \cdot u_1}{\|u_1\|^2} u_1 = (1,-1,0,0) - \frac{1}{2}(1,0,0,-1)$

$= \left(\frac{1}{2}, -1, 0, \frac{1}{2}\right)$

We set $u_2 = \underline{(1,-2,0,1)}$ to avoid fractions.

Now $\tilde{u}_3 = v_3 - \dfrac{v_3 \cdot u_1}{\|u_1\|^2} u_1 - \dfrac{v_3 \cdot u_2}{\|u_2\|^2} u_2$

$\frac{1}{2}$ - vectors are $\perp$

$\frac{1}{2}$ - vectors $\in U$

$= v_3 - \frac{1}{2}(1,0,0,-1) - \frac{1}{6}(1,-2,0,1)$

$= (1,0,1,0) + \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) + \left(-\frac{1}{6}, \frac{1}{3}, 0, -\frac{1}{6}\right)$

$= \left(\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3}\right)$

So set $u_3 = \underline{(1,1,3,1)}$. Then $\{u_1, u_2, u_3\}$ is an orthogonal basis for U.

1 - correct formula

c) $\text{proj}_U (2,0,2,4) = \dfrac{(2,0,2,4)\cdot(1,0,0,-1)}{2}(1,0,0,-1) + \dfrac{(2,0,2,4)\cdot(1,-2,0,1)}{6}(1,-2,0,1)$

$\frac{1}{2}$ $+ \dfrac{(2,0,2,4)\cdot(1,1,3,1)}{12}(1,1,3,1) = -u_1 + u_2 + u_3$

$\frac{1}{2}$ ans $\in U$.

$= (1,-1,3,3)$ $\frac{1}{2}$

13. Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$.

a) Find the characteristic polynomial of A, and use this to show that the eigenvalues of A are 5 and -1.

b) Find a basis of $E_5 = \{v \in \mathbf{R}^3 \mid Av = 5v\}$.

c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If this is not possible, explain why.

a) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} \overset{\text{row2}}{=} -(1+\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} \overset{-R_1+R_3 \to R_3}{=} -(\lambda+1) \begin{vmatrix} 2-\lambda & 3 \\ 1+\lambda & -1-\lambda \end{vmatrix}$

$= -(\lambda+1)^2 \begin{vmatrix} 2-\lambda & 3 \\ 1 & -1 \end{vmatrix} = -(\lambda-2-3)(\lambda+1)^2 = -(\lambda-5)(\lambda+1)^2$. Hence the evals of A are -1 and 5.

b) $E_5 = \ker \begin{bmatrix} -3 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(s,0,s) \mid s \in \mathbb{R}\}$. Hence

$\{\overset{v_1}{(1,0,1)}\}$ is a basis for E_5

c) $E_{-1} = \ker \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-t,s,t)\}$. Hence

$\{\overset{v_2}{(-1,0,1)}, \overset{v_3}{(0,1,0)}\}$ is a basis for E_{-1}.

d) Since $\dim E_{-1} + \dim E_5 = 3$, A is diag'ble. Set

$P = [v_1\ v_2\ v_3] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

block col. form

Then $P^{-1}AP = D$. (P is inv - since $\{v_1, v_2, v_3\}$ is an orthogonal & hence l.i. set!)

14. Let $u = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(v) = u \times v, \qquad \text{for all } v \in \mathbf{R}^3,$$

where "$u \times v$" denotes cross product of u and v. *(You do <u>not</u> have to prove that S is linear.)*

a) If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3$, show that $S(v) = \begin{bmatrix} -2y + z \\ 2x - 2z \\ -x + 2y \end{bmatrix}$.

b) Find a 3×3 matrix A such that $S(v) = Av$, where Av denotes the matrix product of A and v.

c) Find a basis for $\ker S = \{v \mid S(v) = 0\}$, and give a complete geometric description of $\ker S$.

d) Find a basis for $\operatorname{im} S = \{S(v) \mid v \in \mathbf{R}^3\}$, and give a complete geometric description of $\operatorname{im} S$.

a) $S(v) = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ x & y & z \end{vmatrix} = (z-2y, -(2z-2x), 2y-x)$, as req'd.

b) If $A = [Se_1 \;\; Se_2 \;\; Se_3] = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$, then $Sv = Av$.

c) We know
$\ker S = \ker A = \ker \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$= \operatorname{span}\{(1, \frac{1}{2}, 1)\}$, so $\{(2,1,2)\}$ is a basis for $\ker S$.
This is the line through O with direction $(2,1,2)$.

d) We know $\operatorname{im} S = \operatorname{col} A = \operatorname{span}\{\overset{v_1}{(0,2,-1)}, \overset{v_2}{(-2,0,2)}\}$ from previous work. This is the plane through O with normal $(2,1,2)$ (from geometric means, or by computing $n = v_1 \times v_2$).

15. a) State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

i) Suppose C is an invertible 3×3 matrix, and that $\{v_1, v_2, v_3\}$ is linearly independent in $\mathbb{R}^3$. Then $\{Cv_1, Cv_2, Cv_3\}$ is also linearly independent.

Let $A = [v_1 \ v_2 \ v_3]$ (v_j = jth col of A). Then A is invertible, so

CA is too. But $CA = C[v_1 \ v_2 \ v_3] = [Cv_1 \ Cv_2 \ Cv_3]$.

So the cols of CA are l.i.

ANSWER $\boxed{\text{TRUE.}}$

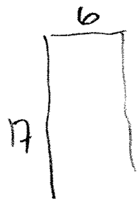
ii) $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable.

The eigenvalues of $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ are 3 and 0 (since $\begin{vmatrix} 3-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda(\lambda-3)$.), which are distinct. Hence $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ is diag'ble.

ANSWER $\boxed{\text{TRUE}}$

15 a) (cont.)

iii) If the columns of a 17×6 matrix A form an orthogonal set in $\mathbf{R}^{17}$, then rank $A = 6$.



$A = [c_1\ c_2 \cdots c_6]$. Since $\{c_1, \ldots, c_6\}$ is orthogonal, $\{c_1, c_2, \ldots, c_6\}$ is l.i. Hence the cols of A are l.i. so rank $A = 6$ $(= \#\text{ cols})$

ANSWER $\boxed{\text{TRUE}}$

15 b). Let A be a $n \times n$ matrix with real entries. Give three additional statements equivalent to

" A is **not** invertible"

one each in terms of

(i) the homogeneous linear system $Ax = 0$:

$Ax = 0 \Rightarrow x = 0$

(ii) the rank of A:

rank $A = n$

(iii) the determinant of A:

$\det A \neq 0$

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

In parts (a) and (b), A denotes a symmetric 100×100 matrix.

a) Prove that $(Au) \cdot u' = u \cdot (Au')$ for all $u, u' \in \mathbf{R}^{100}$, where "$\cdot$" denotes the dot product.

$$\underbrace{(Au) \cdot u'}_{\text{dot}} = \underbrace{(Au)^t u'}_{\text{matrix product}} = u^t A^t u' = u^t A u' = u \cdot (Au').$$

Now let $v_\lambda \in \mathbf{R}^{100}$ be an eigenvector of A with eigenvalue λ, and set $W = \{w \in \mathbf{R}^{100} \mid w \cdot v_\lambda = 0\}$.

b) Prove that if $w \in W$, then $Aw \in W$. Let $w \in W$.

By A, $(Aw) \cdot v_\lambda = w \cdot (Av_\lambda) = w \cdot (\lambda v_\lambda) = \lambda (w \cdot v_\lambda)$
$= \lambda \cdot 0$
$= 0.$

Hence $(Aw) \cdot v_\lambda = 0$, ie $Aw \in W$.