MATHEMATICS 205

Tutorial Assignment 1

The purpose of this tutorial assignment is to give a clearer picture of the idea behind the Riemann integral as a measure of the area under a curve. Later in the course, we will see the Fundamental Theorem of Calculus, which gives a powerful method for finding these areas, but first we will examine the underlying idea.

Example: Estimate the area under the curve $f(x) = x^2 - x + 1$ between $x = 1$ and $x = 9$, using four subdivisions.

Solution:

Step 1: Make a <u>large</u>, careful graph of $y = x^2 - x + 1$ on the interval (1,9).

Step 2: Divide the interval (1,9) into $n = 4$ equal subdivisions, each subinterval having width $\Delta x = (9 - 1)/4 = 2$, with subdivision points $x = 3$, $x = 7$, and $x = 9$. Make a table showing the intervals, their left endpoints x_L, midpoints x_M, and right endpoints x_R, and the values of $f(x)$ at each of these points:

Interval	x_L	$f(x_L)$	x_M	$f(x_M)$	x_R	$f(x_R)$
1	1	1	2	3	3	7
2	3	7	4	13	5	21
3	5	21	6	31	7	43
4	7	43	8	57	9	73

Step 3: For each subinterval, draw rectangles with base on the x-axis and height $f(x_L)$, $f(x_M)$, and $f(x_R)$ respectively. The total area of the rectangles drawn with base Δx and height $f(x_L)$ is

$$\mathbf{A}_L = 1 \times 2 + 7 \times 2 + 21 \times 2 + 43 \times 2 = 144.$$

Similarly, the total area of the rectangles with height $f(x_M)$ and base Δx is

$$\mathbf{A}_M = 3 \times 2 + 13 \times 2 + 31 \times 2 + 57 \times 2 = 208,$$

and the total area of the rectangles with height $f(x_R)$ and base Δx is

$$\mathbf{A}_R = 7 \times 2 + 21 \times 2 + 43 \times 2 + 73 \times 2 = 288.$$

Since $f(x)$ is increasing on the interval (1,9), we have $\mathbf{A}_L < \mathbf{A}_M < \mathbf{A}_R$. Clearly $\mathbf{A}_L$ is less than the area under the curve, while $\mathbf{A}_R$ is greater than the area under the curve. It appears the $\mathbf{A}_M$ is fairly close to the area under the curve.

Step 4: Another way we could get a fairly good approximation to the area under the curve would be to take the average of $f(x_L)$ and $f(x_R)$, that is, $\dfrac{f(x_L) + f(x_R)}{2}$ for the height of our rectangles. This is equivalent to taking trapezoids whose tops runs from $(x_L, f(x_L))$ to $(x_R, f(x_R))$. The total area of these trapezoids is

$$A_T = \frac{1+7}{2}\times 2 + \frac{7+21}{2}\times 2 + \frac{21+43}{2}\times 2 + \frac{43+73}{2}\times 2 = 212.$$

Since $f(x)$ is concave up on (1,9), the tops of the trapezoids all lie above the curve, so $\mathbf{A}_T$ is greater than the area under the curve. If we look closely at the rectangles occurring in $\mathbf{A}_M$, we see that the part of the rectangle to the left of the midpoint and above the curve seems to be less than the part to the right of the midpoint and below the curve. So it appears that $\mathbf{A}_M$ is less than the area under the curve. Another way of seeing this is to draw tangent lines to the curve at the midpoints of each subinterval. These tangent lines lie below the curve (since the curve is concave up), and they form trapezoids with the sides of the rectangles whose areas are equal to the areas of the rectangles with height $f(x_M)$. Putting all this together, we have (since $f(x)$ is increasing and concave up)

$$\mathbf{A}_L < \mathbf{A}_M < \text{area under curve} < \mathbf{A}_T < \mathbf{A}_R.$$

(Note: The obvious thing to do at this stage is to take the average of $\mathbf{A}_M$ and $\mathbf{A}_T$ to get a still better approximation to the area under the curve. It turns out that this is not quite the best thing to do. We leave the subject at this point. If you are interested, you might want to look up “Simpson’s Rule” in the book or online to find out what the best thing is. **Simpson’s Rule is not part of the course, and will not appear on the midterm exam nor on the final exam.**)

Exercise 1: Repeat the above procedure for $f(x) = x^2 - x + 1$ on the interval (1,9), using $n = 8$ subintervals, with $\Delta x = (9 - 1)/8 = 1$. To help you, here is a table that you can fill in:

Interval	x_L	$f(x_L)$	x_M	$f(x_M)$	x_R	$f(x_R)$
1						
2						
3						
4						
5						
6						
7						
8						

Calculate $\mathbf{A}_L$, $\mathbf{A}_M$, $\mathbf{A}_R$, and $\mathbf{A}_T$. What do you observe?

(Note: Your calculator can help you calculate Riemann sums. To find out how, either go to learning.concordia.ca , then click on “Math and Engineering Help”, then on “Scientific Calculator Manual”, and finally on “Getting the Most Out of your Scientific Calculator”, or go directly to

http://cdev.concordia.ca/documents/Learning/ScientificCalc.pdf .

Then go to pages 19-23 (for the Sharp calculator) or 58-62 (for the Casio calculator). You will see how your calculator can do the dirty work of calculating and storing $\mathbf{A}_L$, $\mathbf{A}_M$, $\mathbf{A}_R$, and $\mathbf{A}_T$ for you, without your having to write down a single numerical value or intermediate result.)

(Exercises 2, 3, 4 on the next page.)

Exercise 2: Repeat the above procedure for $f(x) = x^2 - x + 1$ on the interval (1,9), using n subintervals, with $\Delta x = (9 - 1)/n$, where n is an arbitrary positive integer. To help you, here is a table that you can fill in:

Interval	x_L	$f(x_L)$	x_M	$f(x_M)$	x_R	$f(x_R)$
1	1		$1+\frac{4}{n}$		$1+\frac{8}{n}$	
2	$1+\frac{8}{n}$		$1+\frac{12}{n}$		$1+\frac{16}{n}$	
…	…		…		…	
i	$1+\frac{8(i-1)}{n}$		$1+\frac{8(2i-1)}{2n}$		$1+\frac{8i}{n}$	
…	…		…		…	
n	$1+\frac{8(n-1)}{n}$		$1+\frac{8(2n-1)}{2n}$		9	

Express $\mathbf{A}_R$ as a summation with respect to i, and simplify it as much as you can. Then use the formulas on Page A37 (Appendix E) to express the sum as a function of n (not involving i). Finally, take the limit as $n \to \infty$ to find the area under the curve. (**Optional:** Do the same for $\mathbf{A}_L$, $\mathbf{A}_M$, and $\mathbf{A}_T$. The algebra is messy, but the results are interesting.)

Exercise 3: Repeat the procedure of the **Example** at the beginning of this assignment for the interval (1,9) and $n = 4$ for each of the following functions:

(a) $f(x) = \sqrt{x}$ (increasing, concave down)

(b) $f(x) = \frac{1}{x}$ (decreasing, concave up)

(c) $f(x) = \sqrt{100 - x^2}$ (decreasing, concave down)

What do you notice about the relative sizes of $\mathbf{A}_L$, $\mathbf{A}_M$, $\mathbf{A}_R$, and $\mathbf{A}_T$?

Exercise 4 (optional): See what happens when you go from $n = 4$ to $n = 8$ in Exercise 3. How much improvement does there seem to be?