CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Sections**
Mathematics	205	All
Examination	**Date**	**Pages**
Final	December 2009	2
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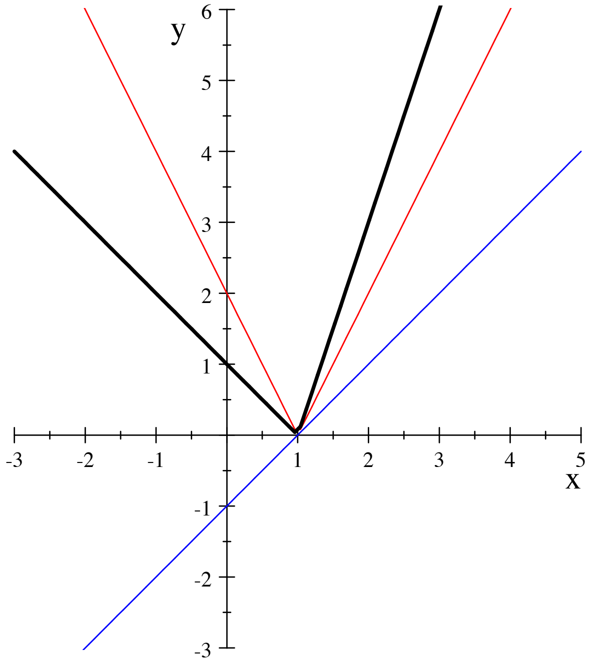
Special Instructions:
- ▷ Materials allowed — Only approved calculators
- ▷ Time allowed — 3 hours
- ▷ Marks for questions are indicated

1 [10]. **(a)** Sketch a graph of the function $f(x) = 2\,|1 - x| + x - 1$, and calculate the definite integral $\int\limits_0^3 f(x)\,\mathrm{d}x$.

(b) Find the derivative of the function $F(x) = \int\limits_0^{2x} \cos(t)\,\sqrt{1+t}\,\mathrm{d}t$ and calculate its value $F\,'(0)$. (HINT: *do not try to integrate*)

Solutions

Solution (a) Start with $2\,|1-x| = 2\,|x-1|$ (red graph) and $x-1$ (blue graph). Then we see that if $x - 1 \geq 0$, i.e. $x \geq 1$, $f(x) = 2(x-1) + x - 1 = 3(x-1)$. But if $x-1 < 0$, i.e. $x < 1$, $f(x) = -2(x-1) + x - 1 = -(x-1)$. So the graph of $f(x)$ which is the sum of the red and blue is the thick graph:



To find $\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx$ we simply calculate the areas of two triangles: $\frac{1}{2}\cdot 1\cdot 1+\frac{1}{2}\cdot 2\cdot 6 = 6.5$

(b) If $F(x) = \int_0^{2x}\cos(t)\sqrt{1+t}dt$ then we can think of $F(x) = G(2x)$ where $G(x) = \int_0^x \cos(t)\sqrt{1+t}dt$ and so

$$\begin{aligned} F'(x) &= G'(2x)\cdot 2 \\ &= \left(\cos(2x)\sqrt{1+2x}\right)\cdot 2 \end{aligned}$$

by the Fundamental Theorem. Substituting $x=0$ gives $F'(0) = 2\cos(0)\sqrt{1} = 2$.

2 [20]. Find the following indefinite integrals:

(a) $\displaystyle\int \frac{(2-\sqrt{x})^2}{x}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{x+3}{x^2-2x-3}\mathrm{d}x$

(c) $\displaystyle\int \sin^5(x)\,\mathrm{d}x$ (d) $\displaystyle\int \sec^2(x)\sqrt{1+\tan(x)}\,\mathrm{d}x$

Solution (a) $\displaystyle\int \frac{(2-\sqrt{x})^2}{x}dx = \int \frac{4-4\sqrt{x}+x}{x}dx = \int \frac{4}{x}dx - 4\int x^{-1/2}dx + \int dx = 4\ln|x| - 8x^{1/2} + x + C$

(b) $\displaystyle I = \int \frac{x+3}{x^2-2x-3}dx = \int \frac{x+3}{(x-3)(x+1)}dx.$ Now by the theory of partial fractions we know that

$$\begin{aligned}\frac{x+3}{(x-3)(x+1)} &= \frac{A}{x-3} + \frac{B}{x+1}\\ &= \frac{A(x+1)+B(x-3)}{(x-3)(x+1)} \text{ and so}\\ x+3 &= A(x+1)+B(x-3)\end{aligned}$$

Put $x = 3$ to get $6 = 4A \Rightarrow A = 3/2$; now put $x = -1$ to get $2 = -4B \Rightarrow B = -1/2$. The integral becomes

$$I = \frac{3}{2}\int \frac{dx}{x-3} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{3}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C$$

(c) $\displaystyle I = \int \sin^5(x)\,dx = \int \sin^4(x)\sin x dx = \int \left(1-\cos^2(x)\right)^2 \sin x dx.$ Now put $u = \cos x$ and $du = -\sin x dx$ to get

$$\begin{aligned} I &= -\int \left(1-u^2\right)^2 du = -\frac{1}{5}u^5 + \frac{2}{3}u^3 - u + C\\ &= -\frac{1}{5}\cos^5(x) + \frac{2}{3}\cos^3(x) - \cos(x) + C\end{aligned}$$

(d) $I = \int \sec^2 x\sqrt{1+\tan x}dx$. We put $u = 1+\tan x$; $du = \sec^2 x dx$ and get

$$\begin{aligned} I &= \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(1+\tan x)^{3/2} + C \end{aligned}$$

3 [10]. Evaluate the following definite integrals (give the exact answers):

(a) $\displaystyle\int_0^1 (x^2 - 2x + 1)\, e^x \, dx$ (b) $\displaystyle\int_0^{\pi/4} \frac{\cos(2t)}{1+\sin^2(2t)}\, dt$

Solution (a) First consider $I = \displaystyle\int (x^2-2x+1)\, e^x dx$ - we do this by parts:

$$\begin{array}{cc} u = x^2 - 2x + 1 & dv = e^x dx \\ du = 2x - 2 & v = e^x \end{array}$$

so $I = (x^2-2x+1)\, e^x - \int (2x-2)\, e^x dx$ and if $J = \int (2x-2)\, e^x dx$ we integrate by parts again: $U = 2x-2$; $dV = e^x dx$; $dU = 2dx$ and $V = e^x$ to get $I = (x^2-2x+1)\, e^x - (2x-2)\, e^x + 2e^x = (x^2-4x+5)\, e^x$ (no $+C$ because we will evaluate from 0 to 1). So $\displaystyle\int_0^1 (x^2-2x+1)\, e^x dx = (x^2-4x+5)\, e^x\big|_0^1 = 2e-5$

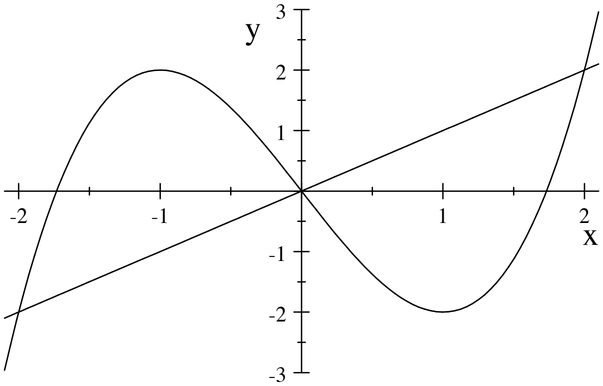
(b) First consider $\int \dfrac{\cos(2t)}{1+\sin^2(2t)}dt$ and let $u = \sin(2t)$; $du = 2\cos(2t)\, dt$. The integral becomes $\dfrac{1}{2}\int \dfrac{du}{1+u^2} = \dfrac{1}{2}\arctan u$ and when the limits are put back we have $\dfrac{1}{2}\arctan(\sin 2t)\Big|_0^{\pi/4} = \dfrac{1}{2}(\arctan 1 - \arctan 0) = \dfrac{\pi}{8}$.

4 [15]. **a.** Find the area enclosed by the curves $y = x^3 - 3x$ and $y = x$.

b. Find the volume of a solid obtained by rotating the region bounded by the curve $y = 1 - \cos(x)$ and the line $y = 1$ on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ about the x-axis.

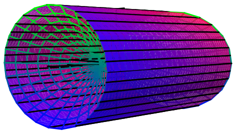
c. Find the mean value of the function $f(x) = \frac{1}{\sqrt{4-x^2}}$ on the interval $[-1, 1]$.

(a) A sketch helps



We see that by symmetry we only need to consider the right half and multiply by 2: $A = 2\int_0^2 (x - (x^3 - 3x))\,dx = 2\int_0^2 (x - x^3 + 3x)\,dx = 2\left(\frac{x^2}{2} - \frac{x^4}{4} + 3\frac{x^2}{2}\right)\Big|_0^2 = 2\,(2 - 4 + 6) = 8.$

(b) Another graphic.



We see that the disc/washer method works best here: $A(x) = \pi\left(R(x)^2 - r(x)^2\right)$ where $R(x) = 1$, the outside radius and $r(x) = 1 - \cos x$. So the volume is obtained by integrating from $-\pi/2$ to $\pi/2$ or, by symmetry, from 0 to $\pi/2$ and then multiplying by 2:

$$\begin{aligned} V &= 2\pi \int_0^{\pi/2} \left(1 - (1-\cos x)^2\right) dx = 2\pi \int_0^{\pi/2} \left(2\cos x - \cos^2 x\right) dx \\ &= 2\pi \int_0^{\pi/2} 2\cos x dx - \pi \int_0^{\pi/2} (1 + \cos(2x))\, dx = 4\pi \sin x\big|_0^{\pi/2} - \pi \left(x + \frac{\sin(2x)}{2}\right)\Big|_0^{\pi/2} \\ &= 4\pi - \frac{\pi^2}{2} \end{aligned}$$

(c) The mean value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\, dx$. In this case $f(x) = \frac{1}{\sqrt{4-x^2}}$ on $[-1, 1]$ so the mean is

$$\begin{aligned} \frac{1}{1-(-1)} \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}} dx &= \frac{1}{2}\left(\arcsin\left(\frac{x}{2}\right)\right)\Big|_{-1}^{1} \\ &= \frac{1}{2}\left(\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)\right) \\ &= \pi/6 \end{aligned}$$

5 [8]. Evaluate the given improper integral or show that it diverges:

$$\text{(a)} \int_0^1 \frac{dx}{x^{3/4}} \qquad\qquad \text{(b)} \int_1^\infty \frac{dx}{x^{3/4}}$$

Solution (a) $\int_0^1 \frac{dx}{x^{3/4}} = \lim_{a\to 0^+} \int_a^1 x^{-3/4} dx = \lim_{a\to 0^+} 4\, x^{1/4}\big|_a^1 = 4\, x^{1/4}\big|_0^1 = 4$

(b) Let $I = \int_1^\infty \frac{dx}{x^{3/4}}$. We know that a Type I integral $\int_1^\infty \frac{dx}{x^p}$ converges only if $p > 1$ and since $p < 1$, I diverges (we don't have to go through the actual calculations here).

6 [10]. Find the limit of the sequence $\{a_n\}$ or prove that the limit does not exist:

$$\text{(a)}\quad a_n = \frac{\sqrt{3+n+n^2}}{1+3n} \qquad\qquad \text{(b)}\quad a_n = \cos(\sqrt{n+1}-\sqrt{n})$$

Solution (a) We do this the same way as we would $\lim\limits_{x\to\infty} \dfrac{\sqrt{3+x+x^2}}{1+3x}$ because a_n is just what we get if we only allow x to be a whole number n. So

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\sqrt{3+n+n^2}}{1+3n} = \lim_{n\to\infty} \frac{n\sqrt{3/n^2+1/n+1}}{n\,(1/n+3)} \\
&= \lim_{n\to\infty} \frac{\sqrt{3/n^2+1/n+1}}{1/n+3} = \frac{\sqrt{\lim\limits_{n\to\infty} 3/n^2 + \lim\limits_{n\to\infty} 1/n + 1}}{\lim\limits_{n\to\infty} 1/n + 3} \quad \text{(limit laws)} \\
&= \frac{\sqrt{0+0+1}}{0+3} = \frac{1}{3}
\end{aligned}$$

(b) This is a bit more challenging. We need to work inside the cosine function to see that $\sqrt{n+1}-\sqrt{n} \to 0$ and therefore since $\cos(0) = 1$ the limit is 1. Here is how we do it:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \cos\left(\sqrt{n+1}-\sqrt{n}\right) \\
&= \cos\left(\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}\right) \\
&= \cos\left(\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \cos\left(\frac{1}{\lim\limits_{n\to\infty}\left(\sqrt{n+1}+\sqrt{n}\right)}\right) \\
&= \cos(0) = 1
\end{aligned}$$

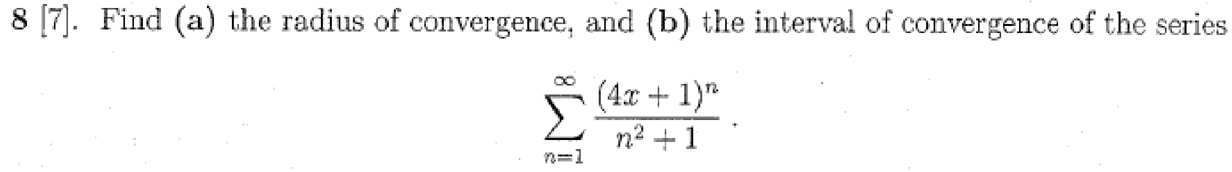
7 [15]. Determine whether the series is divergent or convergent, and if convergent, then absolutely or conditionally :

$$\text{(a)}\quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2+n}} \qquad \text{(b)}\quad \sum_{n=1}^{\infty} \frac{e^n}{2^{n+3}} \qquad \text{(c)}\quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+4n+3}$$

Solution (a)
$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2+n}}$ satisfies the conditions of the alternating series test: $a_n = \frac{1}{\sqrt{2+n}}$ is decreasing (i.e. $a_{n+1} < a_n$) and has limit zero. So the series converges. However $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2+n}}$ diverges by the limit comparison test (compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ with summation starting at 1) or the integral test since $\int_0^\infty \frac{dx}{\sqrt{2+x}}$ diverges (let $u = 2+x$; get $\int_2^\infty \frac{du}{\sqrt{u}} = \int_2^\infty \frac{du}{u^{1/2}}$ which diverges because the exponent p is $1/2$ which is < 1). Therefore this series converges conditionally.

(b) $\sum_{n=0}^{\infty} \frac{e^n}{2^{n+3}} = \frac{1}{2^3} \sum_{n=0}^{\infty} \frac{e^n}{2^n} = \frac{1}{8}\left[1 + \frac{e}{2} + \left(\frac{e}{2}\right)^2 + \cdots\right]$. This is a geometric series with ratio $r = \frac{e}{2} > 1$ and so it diverges.

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+4n+3}$ satisfies the conditions of the alternating series test: $a_n = \frac{1}{n^2+4n+3}$ is decreasing (i.e. $a_{n+1} < a_n$) and has limit zero. So the series converges. Also, since $\frac{1}{n^2+4n+3} < \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ($p-$series with $p = 2$) the given series converges absolutely by the Comparison test.


8 [7]. Find **(a)** the radius of convergence, and **(b)** the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2+1}.$$

Solution (a) We use the ratio test:

$$\begin{aligned} \left|\frac{u_{n+1}}{u_n}\right| &= \frac{|4x+1|^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{|4x+1|^n} \\ &= |4x+1| \frac{n^2+1}{n^2+2n+2} \to |4x+1| \text{ as } n \to \infty \end{aligned}$$

so for absolute convergence we require $|4x+1| < 1$ which is to say $\left|x+\frac{1}{4}\right| < \frac{1}{4}$. Obviously if $|4x+1| > 1$ ($\left|x+\frac{1}{4}\right| > \frac{1}{4}$) the series would diverge. So the radius of convergence is $\frac{1}{4}$.

(b) We know the open interval is given by $-1/2 < x < 0$, i.e. $(-1/2, 0)$. To find the exact interval of convergence we need to test the endpoints.

At $x = -1/2$ we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ which converges absolutely since this is similar to question 7 (c) above,

At $x = 0$ we get $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ which also converges absolutely (by the Comparison test). So the interval of convergence is $[-1/2, 0]$.

9 [5]. Find the radius of convergence of the series and, within this radius, the sum of the series as a function of x:

$$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$$

Solution We see right away that this is a geometric series with first term $a = 1$ and $r = \frac{(x-1)^2}{4}$. We also know that

(i) a geometric series converges only if $|r| < 1$, i.e. $-1 < r < 1$ so the endpoints are automatically excluded. In this case it means (since $(x-1)^2$

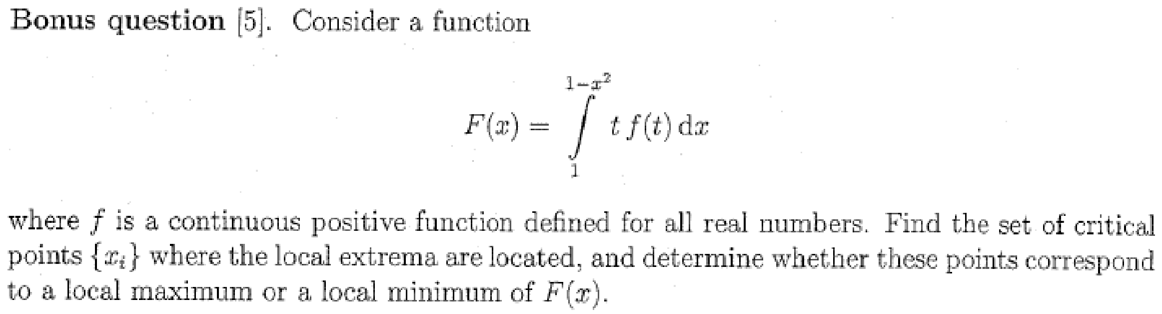
can't be negative)

$$\begin{aligned}\frac{(x-1)^2}{4} &< 1\\ (x-1)^2 &< 4 \Longrightarrow |x-1| < 2 \text{ and so}\\ -1 &< x < 3\end{aligned}$$

The radius of convergence is not 1 (the maximum of r of the geometric series), but the 2 we get from the calculation above.

(ii) The sum of the series is, of course

$$\frac{a}{1-r} = \frac{1}{1-\dfrac{(x-1)^2}{4}} = \frac{4}{3+2x-x^2}$$



Bonus question [5]. Consider a function

$$F(x) = \int_{1}^{1-x^2} t\,f(t)\,\mathrm{d}x$$

where f is a continuous positive function defined for all real numbers. Find the set of critical points $\{x_i\}$ where the local extrema are located, and determine whether these points correspond to a local maximum or a local minimum of $F(x)$.

Solution Note that the dx in the question is a typo - it should be dt. We first find the derivative of $F(x)$, using the Fundamental Theorem and the Chain Rule:

$$\begin{aligned}F'(x) &= \left(1-x^2\right) f\left(1-x^2\right) \frac{d}{dx}\left(1-x^2\right)\\ &= -2x\left(1-x^2\right) f\left(1-x^2\right)\\ &= 2x\left(x^2-1\right) f\left(1-x^2\right)\end{aligned}$$

We see right away that since f is a positive function, the only critical points are $0, -1$ and 1. Since we can't take the second derivative (we don't know if

f is differentiable) we use the first derivative test.

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$2x$	$-$		$-$		$+$		$+$
$x - 1$	$-$		$-$		$-$		$+$
$x + 1$	$-$		$+$		$+$		$+$
$F'(x)$	$-$	0	$+$	0	$-$	0	$+$

We see that there is a local minimum at $x = -1$ and $x = 1$ and a local maximum at $x = 0$.