CONCORDIA UNIVERSITY

Course: Math **Number**: 205/2 **Sections**: all

Examination:Final **Date**:December 2008 **Time**:3hours **#of pages**: 2

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Special instructions: Only approved calculators are allowed.

(**MARKS**)

(**10**) **1**

a. Give lower and upper estimates for the area bounded by the graph of the function $f(x) = x^2$ from $x = 1$ to $x = 5$ using four rectangles. Sketch the graph and the rectangles.

b. Sketch the graph of the function:

$$f(x) = \begin{cases} 2x+2 & for: -2 \le x < -1 \\ \sqrt{1-x^2} & for: -1 \le x < 1 \\ x & for: 1 \le x < 2 \end{cases}$$

Evaluate the definite integral $\int_{-2}^{2} f(x)dx$ by interpreting it in terms of area.

c. Calculate the derivative of the function: $F(x) = \int_{\sqrt{x}}^{5} e^{t^2}\,dt$ (Hint: Do not integrate).

(**15**) **2** Calculate the indefinite integrals:

a. $\int \cos^3 x \sin^2 x dx$

b. $\int \sqrt{4-x^2}\,dx$

c. $\int \frac{xdx}{x^2-3x+2}$ (as it should've been) - here was the misprint: $\int \frac{xdx}{x^2-x+2}$.

(**15**) **3** Calculate the definite integrals:

a. $\int_{2}^{\sqrt{e}} \frac{x^2 - \ln x^5}{x}dx$ (Hint: First simplify the logarithm)

b. $\int_{0}^{4} \sin\frac{\pi\sqrt{x}}{4}dx$

c. $\int_{0}^{1} \frac{x^2-2x+4}{1+x^2}dx$

(**15**) **4** Calculate:

a. the area between curves: $y = 3x^2$ and $x = 3y^2$,

b. the volume of the solid obtained by rotating the region bounded by curve $y = e^x$, lines: $y = -1$, $x = -1$ and $x = 1$ about $y = -1$,

c. the average value of function $f(x) = \frac{\arctan x}{1+x^2}$ on interval $[0,1]$.

(**12**) **5** Determine whether the following integral is convergent or divergent and evaluate those that are convergent.

a. $\int_0^1 \frac{dx}{\sqrt{x}}$

b. $\int_1^\infty \frac{dx}{\sqrt{x}}$

(**9**) **6** Determine the formula for a_n for the sequence: $\left\{\frac{5}{9}, -\frac{15}{25}, \frac{25}{49}, -\frac{35}{81}, \ldots\right\}$ and calculate its limit.

(**12**) **7** For each for the following number series determine whether it is convergent (absolutely or conditionally), or divergent:

a. $\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}$

b. $\sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{n^2+1}$

c. $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

(**12**) **8**

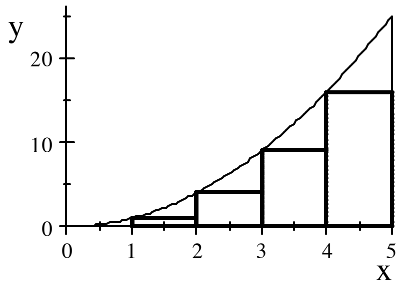
a. Calculate the sum: $\sum_{n=2}^{\infty} (-1)^n \frac{2^n}{3^{2n+1}}$

b. Establish the interval of convergence for the power series: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{\sqrt{n}}$

(**5**) **9** Write the MacLaurin series expansion and determine the interval of convergence for the functions: $f(x) = \arctan x$.

<u>Solutions for the final - Math 205</u>

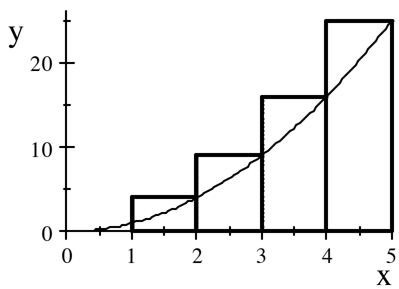
1. **a**. **i**. right end points:



Leftpoin (lower) estimate

$\mathbf{R}_4 = \sum_{i=1}^{4} i^2 = 30$ is an underestimate.

ii. left end points:

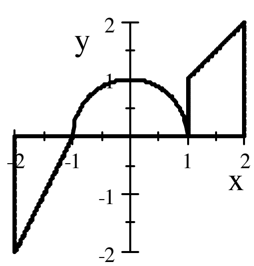


Rightpoint (upper) estimate

$\mathbf{L}_4 = \sum_{i=2}^{5} i^2 = 54$ - is an overestimate.

Therefore the area $A \in (\mathbf{30}, \mathbf{54})$

b. The graph of $y = f(x)$ is:



The $\int_{-2}^{2} f(x)dx = A = -A_1 + A_2 + A_3 =$

$-1 + \frac{\pi}{2} + \frac{3}{2} = \frac{\pi}{2} + \frac{1}{2}$

c. $F'(x) = \frac{d}{dx}\int_{\sqrt{x}}^{5} e^{t^2}dt = \frac{d}{dx}\left(F(5) - F(\sqrt{x})\right) = -\frac{e^x}{2\sqrt{x}}$, using the chain rule and Fundamental Theorem of Calculus.

2. **a.** $\int \cos^3 x \sin^2 x dx = \int \cos x \sin^2 x \left(1 - \sin^2 x\right) dx = \left| \begin{array}{c} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int \left(t^2 - t^4\right) dt =$

$\frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

b. $\int \sqrt{4 - x^2}\, dx = \left| \begin{array}{c} x = 2\sin t \\ dx = 2\cos t dt \\ \sqrt{4 - x^2} = \sqrt{4 - 4\sin^2 t} = 2\cos t \end{array} \right| = \int 4\cos^2 t dt = 2t + 2\sin t \cos t + C = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4 - x^2} + C.$

c. $\int \frac{xdx}{x^2 - 3x + 2} = \int \frac{xdx}{(x-2)(x-1)} = \int \frac{adx}{x-2} + \int \frac{bdx}{x-1}$

Differentiate: $\frac{x}{(x-2)(x-1)} = \frac{a}{x-2} + \frac{b}{x-1}$

Multiply by $(x-2)(x-1)$: $x = a(x-1) + b(x-2)$

For $x = 2 : 2 = a$

For $x = 1 : 1 = -b \to b = -1$

Giving: $\int \frac{xdx}{x^2 - x + 2} = \int \frac{2dx}{x-2} + \int \frac{-1dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C = \ln\left|\frac{(x-2)^2}{x-1}\right| + C.$

3. **a.** $\int_2^{\sqrt{e}} \frac{x^2 - \ln x^5}{x} dx = \int_2^{\sqrt{e}} \left(x - \frac{5\ln x}{x}\right) dx = \frac{x^2}{2} - \frac{5\ln^2 x}{2}\Big]_2^{\sqrt{e}} = \frac{e}{2} + \frac{5}{2}\ln^2 2 - \frac{21}{8}$

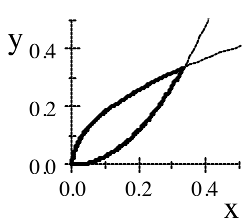
b. $\int_0^4 \sin\frac{\pi\sqrt{x}}{4} dx = \left| \begin{array}{cc} \frac{16t^2}{\pi^2} = x & 0 \to 0 \\ \frac{32tdt}{\pi^2} = dx & 4 \to \frac{\pi}{2} \end{array} \right| = \frac{32}{\pi^2}\int_0^{\frac{\pi}{2}} t\sin t dx =$

$\left| \begin{array}{cc} u = t & du = dt \\ dv = \sin t dt & v = -\cos t \end{array} \right| = \frac{32}{\pi^2}\left(-t\cos t\Big]_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \cos t dt\right) = \frac{32}{\pi^2}$

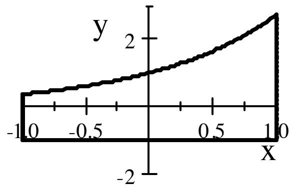
c. $\int_0^1 \frac{x^2 - 2x + 4}{1 + x^2} dx = \int_0^1 \left(1 - \frac{2x}{1 + x^2} + \frac{3}{1 + x^2}\right) dx =$

$x - \ln\left(1 + x^2\right) + 3\arctan x\Big]_0^1 = \frac{3\pi}{4} - \ln 2 + 1$

4. **a.** the area between curves: $y = 3x^2$ & $x = 3y^2 \to y = 27y^4 \to y = 0, \frac{1}{3} \to A = \int_0^{\frac{1}{3}} \left(\sqrt{\frac{y}{3}} - 3y^2\right) dy = \frac{1}{27}.$



b. the volume of the solid obtained by rotating the region bounded by curve $y = e^x$, lines: $y = -1$ and $x = -1$ & $x = 1$ about $y = -1$,



$$V = \pi \int_{-1}^{1} \left(1 + e^x\right)^2 dx = \pi \int_{-1}^{1} \left(1 + 2e^x + e^{2x}\right) dx =$$

$$\pi\left(2 + 2\left(e - \frac{1}{e}\right) + \frac{e^2 - \frac{1}{e^2}}{2} \right) \approx 32.445$$

c. the average value of function $f(x) = \frac{\arctan x}{1+x^2}$ on interval $[0,1] \to f_{ave} == \int_0^1 \frac{\arctan x}{1+x^2} dx =$

$$\left| \begin{array}{ll} t = \arctan x & x = 0 \to t = 0 \\ dt = \frac{dx}{1+x^2} & x = 1 \to t = \frac{\pi}{4} \end{array} \right| = \int_0^{\frac{\pi}{4}} t dt = \frac{\pi^2}{32}$$

5. **a**. The integral: $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0^+} \left(2\sqrt{t} - 2\right) = 2$ is convergent.

b. The integral: $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{t \to \infty} \int_1^t \frac{dx}{\sqrt{x}} = \lim_{t \to \infty} \left(2\sqrt{t} - 2\right) = \infty$ is divergent.

6. The sequence: $\left\{\frac{5}{9},-\frac{15}{25},\frac{25}{49},-\frac{30}{81},\ldots\right\} = \left\{(-1)^{n+1}\frac{5+10n}{(2n+3)^2}\right\}_{n=0}^{\infty}$. The limit is then:

$\lim|a_n| = \lim\frac{5+10n}{(2n+3)^2} = \lim_{x\to\infty}\frac{5+10x}{(5x+3)^2} = 0 \to \lim a_n = 0.$

7. **a**. $\sum_{n=1}^{\infty}\frac{n!}{(2n-1)!}$: Ratio test: $\lim\frac{a_{n+1}}{a_n} = \lim\frac{(n+1)!}{(2n+1)!}\frac{(2n-1)!}{n!} = \lim\frac{n+1}{2n(2n+1)} = 0 < 1 \to$ convergent (absolutely)

b. $\sum_{n=1}^{\infty}(-1)^n\frac{2n-1}{n^2+1}$: $\lim\frac{2n-1}{n^2+1} = 0$ and the function $f(x) = \frac{2x-1}{x^2+1}$ for $x \geq 2$ has derivative: $f'(x) = \frac{2(-x^2+x+1)}{(x^2+1)^2} < 0 \to \frac{2n-1}{n^2+1}$ is decreasing for $n \geq 2 \to$ convergent.

Now compare $\sum_{n=1}^{\infty}\frac{2n-1}{n^2+1}$ to $\sum_{n=1}^{\infty}\frac{1}{n}$ (divergent): $\lim\frac{(2n-1)n}{n^2+1} = 2 \to$ the alternating series is conditionally convergent.

c. $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$: Integral test: $\int_2^{\infty}\frac{dx}{x\ln^2 x} = \frac{1}{\ln 2} < \infty \to$ convergent

Therefore the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\sqrt{n}}$ is conditionally convergent.

8. **a**. Calculate $S = \sum_{n=2}^{\infty}(-1)^n\frac{2^n}{3^{2n+1}} = \frac{1}{3}\sum_{n=2}^{\infty}(-1)^n\frac{2^n}{9^n}$: geometric series with $r = -\frac{2}{9} \to S = \frac{1}{3}\frac{\frac{4}{81}}{1+\frac{2}{9}} = \frac{4}{297}$

b. The radius of convergence for $\sum_{n=1}^{\infty}\frac{(2x+1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{2^n(x+\frac{1}{2})^n}{\sqrt{n}}$: $r = \lim\frac{a_n}{a_{n+1}} = \lim\frac{2^n\sqrt{n+1}}{2^{n+1}\sqrt{n}} = \frac{1}{2}$

The center is $a = -\frac{1}{2} \to$ the interval (without end points) is $(-1,0)$.

At $x = -1$: $\sum_{n=1}^{\infty}\frac{(2(-1)+1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ as $\lim\frac{1}{\sqrt{n}} = 0$ and decreasing, therefore it is convergent;

At $x = 0$: $\sum_{n=1}^{\infty}\frac{(2(0)+1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent as it is a $p-$ series with $p = \frac{1}{2} < 1$.

Interval of convergence is: $[-1,0)$, or $-1 \leq x < 0$.

9. $f(x) = \arctan x \to f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n$, if $|x| < 1 \to \arctan x = \sum_{n=0}^{\infty}(-1)^n\int_0^x t^{2n}dt = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

Endpoints: $x = -1$: $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$: $\lim\frac{1}{2n+1} = 0$ and decreasing, therefore it is convergent;

$x = 1$: $\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$: $\lim\frac{1}{2n+1} = 0$ and decreasing, therefore it is convergent. The interval of convergence: $[-1,1]$.