MA121
Midterm 1 - Review Notes

1.1 Sets

Definition - Set:
A collection of elements. E.g. A = {a,b,c} is a set. Ways to write a set:

1. Listing: {1,2,3...}

2. Describing: $\{x : x^2 = 4\}$

Cardinality of a set:

|S| = number of elements in set S.

E.g. Let A = {1,2,3,5}, |A| = 4
E.g. Let B = {{1,2,3},{5,8},{11,13}}, |B| = 3

1.2 Subsets:

Definition - Subset:

If for every element of A also belongs to B. We write $A \subseteq B$. There are two kinds of subsets:

1. **Proper subset:** $A \subset S$: A can contain any elements of S except for S itself.

2. **Regular subset:** $B \subseteq S$: B can contain any elements of S, including S itself.

1.3.1 Set Operations:

1. Union: denoted by $A \cup B$, is a set of all the elements that belong to either A or B, without repetition.

2. Intersection: denoted by $A \cap B$, is a set of all the common elements that belong to both A and B.

3. Subtraction: denoted by $A-B$, is a set formed by eliminating the common elements of both A and B from A.

4. Cartesian Product: denoted by $A \times B$, is the set of all possible ordered pairs whose first component is a member of X and whose second component is a member of Y:
$X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}$

Set of elements consists of purely basic elements, no sets. While a mixed set can contain elements that are both sets and basic elements. Distinction has to be made while operating on these mixed type of sets:

1. Let A={a,b,c,d,{d}} and B={a,b,c,d}, then A∩B={d}
2. Let A={a,b,c,d,{d}} and B={a,b,c,{d}}, then A∩B={{d}}

1.3.2 Open vs. Closed Set

1. Open Sets (0,1]= {x∈R:0<x≤1}

2. Closed Sets [0,1)= {x∈R:0≤x<1}

1.3.3 Power Set Definition:

A set of all possible subset of A, denoted as P(A)

E.g. Let A = {1,2,3} the power set of A, P(A) = {{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3},∅}

In general, $|P(A)| = 2^{|A|}$, and ∅ is always an element of any powersets.

2.1 Statement: Definition:

A statement is a declarative sentence that is either true, or false.

E.g.: I have three bouncy balls. (p)

Note: p can be either true, or false. The validity of p does not need to be examined for p to be a statement.

Statement ... NOT: What is the meaning of life? (invalid) Truth table for one, two and three statements {see book pg.31}. 2.2 Negation:
Definition:

Negation of statement P is not P. E.g.: I do not have three bouncy balls. (~ p)

2.3 Disjunction and Conjunction:

Definition:

1. Disjunction: denoted by P ∨ Q, means P OR Q. If either true, then disjunction true.

2. Conjunction: denoted by $P \wedge Q$ means P AND Q. Conjunction true if and only if both P, Q are true.

Tables for disjunction and conjunction - see book pg. 32, 33. 2.4 Implication:

If P then Q: $P \Rightarrow Q$

Remember the TRUTH TABLE on page 33!!

The situation for which $P \Rightarrow Q$ is false when P is true and Q is false, alternatively denoted as $\sim (P \Rightarrow Q)$, which is equivalent to $P \wedge (\sim Q)$.

<u>2.6 Biconditional:</u>

Definition: $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, equivalent to $P \Leftrightarrow Q$. Remember the truth table on pg. 37.

Ref: to Chapter 3: in a proof, if bi-conditional is concerned, then you have to prove both directions.

<u>2.7 Tautologies and Contradictions:</u>

Definition 1: A compound statement S is a tautology if it is true for all possible combinations of truth values of component statements that compose S.

In other words, no matter what values the statements take, the values of S will always be the same. Eg. $P \vee (\sim P)$, $(\sim Q) \vee (P \Rightarrow Q)$, etc.

Definition 2: A compound statement S is a contradiction if it is false for all possible combinations of truth values of component statements that compose S.

Eg. $P\wedge(\sim P)$,$(P\wedge Q)\wedge(Q\Rightarrow\sim P)$,etc.
All tautologies and contradictions are provable by truth-table. Writing down every part of the statement

will help you to be clear about the process. 2.8 Logical Equivalence:

Two statements are logically equivalent when they produce the same results on the truth table with same given conditions. See pg. 39 for table showing $P \Rightarrow Q$ and $(\sim P) \vee Q$

All equivalences can be shown on truth tables. Definition: four fundamental equivalences:

1. **Commutative:** $P\vee Q$ is equivalent to $Q\vee P$, $P\wedge Q$ is equivalent to $Q\wedge P$

2. **Associative:** P∨(Q∨R) is equivalent to (P∨Q)∨R, P∧(Q∧R) is equivalent to (P∧Q)∧R

3. **Distributive:** P∨(Q∧R) is equivalent to (P∨Q)∧(P∨R), P∧(Q∨R) is equivalent to (P∧Q)∨(P∧R)

4. **DeMorgan's:** ~(P∨Q)is equivalent to(~P)∧(~Q),~(P∧Q)is equivalent to (~P)∨(~Q)

2.11 Quantified Statements:

∃ - there exists / for some, aka existential quantifier. ∀ - for all, aka universal quantifier. E.g. A quantified statement: if $x^2 > 0$ then $x > 0$ (this is not always a true statement)

Note: The negation of ∃ is ∀

E.g.: $\forall x \in R, \exists y$, such that $y > x$.

3.1 Vacuous Proofs/ Trivial Proofs:

A vacuous proof is a proof where the condition is intrinsically a false statement, causing the statement to be true automatically.

E.g. If $x^2 + 2x + 2 \leq -1$ then prove $x^3 > 8$.

Proof: we know that $x^2 + 2x + 2 = (x^2 + 2x + 1) + 1 = (x+1)^2 + 1$. Since $(x+1)^2$ is always greater than 0, therefore $x^2 + 2x + 2 \leq -1$ renders false. Therefore $x^3 > 8$ proven vacuously.

A Direct is a proof that the conclusion Q can be directly derived from condition P. E.g. If x is odd then prove $3x-1$ is even.

Proof: x is odd $\Rightarrow x = 2k+1, k \in Z$. Therefore $3x-1 = 3(2k+1)-1 = 6k+3-1 = 6k-2 = 2(3k-1)$. Let some $m \in Z$, $m = 3k - 1$, we have $3x - 1 = 2m \Rightarrow 3k-1$ is even.

3.2 Contra-positive Proofs:

Recall from Chapter 2 that contrapositive means $P \Rightarrow Q$ and $\sim P \Rightarrow \sim Q$ are logically equivalent, we therefore aim to prove $P \Rightarrow Q$ by proving $\sim P \Rightarrow \sim Q$. E.g. x^2 is even only when x is even.

Proof: "x^2 is even"is our P, and"x is even"is Q. So to prove $P \Rightarrow Q$, we prove $\sim P \Rightarrow \sim Q$ or,"x is odd" then "x^2 is odd".

Assume now x is odd, $x = 2k+1$, then $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, let $2k^2 + 2k = m$, we see that $x^2 = 2m + 1$. Therefore obvious x^2 is odd.

Thus by contrapositive, x^2 is even only when x is even.

3.3 If and Only If (biconditional) Proofs:

Recall from Chapter 2 that biconditional means $P \Leftrightarrow Q$ or alternatively, $P \Rightarrow Q$ and $Q \Rightarrow P$.

To prove a biconditional ("if and only if") statement, we need to break it down to two smaller proofs: the "if", and the "only if". Then prove them separately.

E.g. x^2 is even if and only if x is even.

Proof:

(i) the "if" part ($\Rightarrow$):

Suppose x is even, then $x = 2k$, $k \in Z$. $x^2 = (2k)^2$ or $4k^2 = 2(2k^2)$, which is clearly even. (ii) the "only if" part ($\Leftarrow$):

(Apply the contrapositive proof given by previous example here and prove this part.)

3.4 Case-wise Proofs:

Sometimes we need to make cases of the statements to be proven in order to sufficiently prove something. Especially when the conclusion that we try to prove is a vaguely defined one.

E.g. Prove that $n^2 + 3n + 5$ is odd.

Proof: (by cases)

case i: n is even (substitute $n = 2k$, then proceed to show that $n^2 + 3n + 5$ is odd. case i: n is odd (substitute $n = 2k + 1$, then proceed to show that $n^2 + 3n + 5$ is odd.

Since n can only be either even or odd, if both cases prove that $n^2 + 3n + 5$ is odd, then we are done.