

CARLETON UNIVERSITY

FINAL EXAMINATION SOLUTIONS
MATH 2004 A, B, C, D
Fall 2013

DURATION: 3 HOURS

Department Name and Course Number: School of Mathematics and Statistics, MATH 2004 A, B, C, D.
Course Instructor(s): Dr. A.B. Mingarelli (Sect. A), Dr. R. Cova (Sect. B, C), Dr. E. Xhua (Sect. D)

AUTHORIZED MEMORANDA
NON-PROGRAMMABLE CALCULATOR PERMITTED.
MANY **BLANK SHEETS OF PAPER** FOR ROUGH WORK

This exam may be released to the Library and **MAY NOT BE TAKEN AWAY BY THE STUDENT.**

- Please verify that you are in possession of a SCANTRON FORM
- Please fill in your COURSE CODE (e.g., MATH 2004) and COURSE SECTION (e.g., A, B, C, D), as per above list of instructors; YOUR NAME and YOUR STUDENT NUMBER where required on the Scantron form AND on this examination.
- **The entire examination consists of 5 pages and two parts, A and B, and is marked out of a total of 80, that is, 40% of your final mark.** Part A consists of 12 multiple choice questions each worth 3 marks. **Please fill in only one answer on your Scantron sheets with a pencil** as there is only one answer to any given question. Circling two or more answers to any question invalidates that question (*i.e.*, you get 0 marks for that question). Part B consists of 6 traditional type questions (with explanations required), for a total of 44 marks for that section. **Both Part A and B must be submitted along with the scantron sheet, so do not detach nor unstaple this examination. Do not submit rough work.**

Print Name : _____

Student Number: _____

Section (either A, B, C, or D): _____

- **A1.** Find the slope of the tangent line to $r = 3 \sin \theta$ at $\theta = \frac{\pi}{2}$.

(a) 0 (b) $\sqrt{3}$ (c) 2 (d) 3

Solution: (a)

- **A2.** Let θ be the angle between the two planes $x + 2y - 2z - 3 = 0$ and $2x + 2y - z - 5 = 0$. Find $\cos \theta$.

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{8}{9}$ (d) $\frac{9}{8}$

Solution: (c)

- **A3.** Find the area of the triangle with vertices $P(1, 2, 3)$, $Q(-3, 0, 1)$, and $R(2, 4, 5)$.

(a) $6\sqrt{2}$ (b) $6\sqrt{3}$ (c) 12 (d) $3\sqrt{2}$

Solution: (d)

Print Name and Student Number: _____

- **A4.** Given the line $L : \mathbf{r} = (1, 2, 3) + t(3, 2, -1)$. Let Π be the plane through the point $(1, 1, 3)$ and perpendicular to the line L . Which of the following is a point on the plane Π ?

(a) $(0, 0, -1)$ (b) $(0, 0, -2)$ (c) $(0, 0, -3)$ (d) $(0, 0, -4)$

Solution: (b)

- **A5.** Let $w(x, y) = x^2 + y + 3xy^4$, where $x = \sin 2t, y = \cos t$. Use the Chain Rule to find the value of $\frac{\partial w}{\partial t}$ at $t = 0$:

(a) 0 (b) 6 (c) 3 (d) 2

Solution: (b)

- **A6.** A vector giving the direction in which $f(x, y, z) = \frac{x}{y} + \frac{y}{z}$ increases most rapidly at the point $(1, -1, 2)$ is given by

(a) $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ (b) $-\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$ (c) $2\mathbf{i} - \frac{1}{4}\mathbf{j} - \frac{1}{2}\mathbf{k}$ (d) $-\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{4}\mathbf{k}$

Solution: (d)

- **A7.** Find all the critical points of $h(x, y) = y^2 - 2y - y\sqrt{x} + \frac{x}{2}$ where x, y are real, $x \geq 0$.

(a) $(4, 2), (0, 0)$ (b) $(0, 0)$ only (c) $(2, 4), (0, 0)$ (d) $(4, 2)$

Solution: (a)

- **A8.** Evaluate the double integral $\iint_{\mathcal{R}} e^{-x} \sin y \, dx \, dy$, where $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq \ln 2, 0 \leq y \leq \pi/2\}$.

(a) $1 - e$ (b) $\ln 2 - 1$ (c) 0 (d) $\frac{1}{2}$

Solution: (d)

- **A9.** Evaluate the triple integral $\iiint_{\mathcal{T}} xyz \, dz \, dy \, dx$ where \mathcal{T} is the region

$\mathcal{T} = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, \sqrt{x^2 + y^2} \leq z \leq 2\}$.

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{8}$ (d) $\frac{3}{2}$

Solution: (c)

- **A10.** Which one of the following integrals is equal to the double integral $\iint_{\mathcal{R}} 2xy \, dA$, under the change of variables

$x = 3u, y = 4v$ where \mathcal{R} is the elliptical region $16x^2 + 9y^2 \leq 144$.

(a) $\iint_{\mathcal{S}} 12uv \, dA$ (b) $\iint_{\mathcal{S}} 288uv \, dA$ (c) $\iint_{\mathcal{S}} 8(u^2 + v^2) \, dA$ (d) $\iint_{\mathcal{S}} 12u^2v^2 \, dA$

where $\mathcal{S} = \{(u, v) : u^2 + v^2 \leq 1\}$.

Solution: (b)

Print Name and Student Number: _____

- **A11.** Evaluate the line integral $\int_{\mathcal{C}} (6xy^2 - y^3) dx + (6x^2y - 3xy^2) dy$ where \mathcal{C} is the straight line from the point $(1, 2)$ to $(3, 4)$.

(a) 236 (b) 0 (c) 188 (d) 48

Solution: (a)

- **A12.** Let $\mathbf{F}(x, y, z) = 7x\mathbf{i} - z\mathbf{k}$. Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS$ where \mathcal{S} is the surface of the sphere $x^2 + y^2 + z^2 = 4$ and \mathbf{n} is an outer normal unit vector to \mathcal{S} .

(a) 24π (b) 16π (c) 0 (d) 64π

Solution: (d)

End of Part A

Print Name and Student Number: _____

PART B: Do All Four (6) Questions for a total of 44 marks out of a maximum of 80.

Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **B1.** [8 marks] Using the method of Lagrange multipliers determine the local extrema and saddle points (if any) of $f(x, y, z) = x + 2y - 2z$ subject to the constraint $x^2 + 2y^2 + 4z^2 = 1$.

Solution: $\nabla f = \lambda \nabla g$, $g = x^2 + 2y^2 + 4z^2 \rightarrow \lambda = \pm 1$. The critical points are then $CP1(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4})$, $CP2(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4})$. One finds that $f(CP1) = 2$ and $f(CP2) = -2$ are the local maximum and minimum, respectively.

- **B2.** [7 marks] By changing to polar coordinates evaluate the integral $\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2) dx dy$.

Solution: $\int_0^{\pi/2} \int_0^1 \cos(r^2) r dr d\theta = \frac{\pi}{4} \sin(1)$.

Print Name and Student Number: _____

- **B3.** [7 marks] The parametric curve C in the plane is given by $x = 2t^2$, $y = 1 + t^3$, where $0 \leq t \leq 1$. Find the length of the curve C .

Solution: We have $x'(t) = 4t$, $y'(t) = 3t^2$. The length is

$$\begin{aligned} L &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} dt \\ &= \int_0^1 \sqrt{16t^2 + 9t^4} dt = \int_0^1 t\sqrt{16 + 9t^2} dt \\ &= \frac{1}{18} \int_{16}^{25} \sqrt{u} du, \quad u = 16 + 9t^2, \\ &= \frac{1}{27} u^{3/2} \Big|_{16}^{25} = \frac{61}{27}. \end{aligned}$$

- **B4.** [7 marks] Find the area of the region enclosed by the cardioid $r = 1 + \sin \theta$ and the circle $r = 3 \sin \theta$.

Solution: The intersection points of the two curves are given by $1 + \sin \theta = 3 \sin \theta$, from which we get $\sin \theta = 1/2$ or $\theta = \pi/6$ or $5\pi/6$. The area is then given by

$$\begin{aligned} A &= \int_{\alpha}^{\beta} \frac{1}{2} (r_0^2 - r_1^2) d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 \sin^2 \theta - (1 + \sin \theta)^2) d\theta \\ &= \pi. \end{aligned}$$

Print Name and Student Number: _____

- **B5.** [7 marks] Let $\mathbf{F}(x, y, z) = \frac{1}{yz} \mathbf{i} - \frac{x}{y^2z} \mathbf{j} - \frac{x}{yz^2} \mathbf{k}$.

a) Compute $\text{curl } \mathbf{F}$.

b) Find a scalar field $f(x, y, z)$ such that $\nabla f = \mathbf{F}$. If no such scalar field exists, explain why this is not possible.

Solution:

a) $\text{curl } \mathbf{F} = \mathbf{0}$.

b) $f(x, y, z) = \frac{x}{yz}$.

- **B6.** [8 marks] Let $\mathbf{F}(x, y, z) = -5y \mathbf{i} + 4x \mathbf{j} + z \mathbf{k}$ and let \mathcal{S} be the surface bounded by the circle $x^2 + y^2 = 4$ and $z = 1$. Evaluate the surface integral $\iint_{\mathcal{S}} \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS$ using any method. Here \mathbf{n} is an outer normal unit vector to \mathcal{S} .

Solution:

Direct calculation gives that $\text{curl } \mathbf{F} = 9\mathbf{k}$. In addition, because the normal lies on the plane $z = 1$ it is clear that $\mathbf{n} = \mathbf{k}$. Hence $\text{curl } \mathbf{F} \cdot \mathbf{n} = 9$. Next the region \mathcal{R} in the uv -plane is a circle of radius 2. It follows that

$$\begin{aligned} \iint_{\mathcal{S}} \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS &= 9 \iint_{\mathcal{R}} du \, dv \\ &= 9 \cdot \text{Area of } \mathcal{R} \\ &= 9\pi 2^2 = 36\pi. \end{aligned}$$