

CARLETON UNIVERSITY

FINAL
EXAMINATION
Fall 2010

DURATION: 3 HOURS

Department Name and Course Number: School of Mathematics and Statistics, MATH1104.
Course Instructors: J. Abdulrahman, A. Sakzad, L. Bourbonnais, and M. Lemire.

AUTHORIZED MEMORANDA

NON-PROGRAMMABLE CALCULATOR PERMITTED

This examination paper **MAY NOT** be taken from the examination room.
Scratch paper is not permitted, you can write on the backs of the pages if needed.

First (Given) Name :
Student No :

Last Name:

PLEASE READ THE FOLLOWING INSTRUCTIONS BEFORE PROCEEDING

1. Please count your pages now. **This examination has 10 pages**, including this page. If you feel there is a page missing, please report this to your Proctor.
2. **The examination is out of a total of 100 marks** and consists of 17 questions. There are 12 multiple choice questions worth 4 points each and 5 long answer questions. **Place your answers to the multiple choice questions in the boxes on the second page.** For the long answer questions **you must show all your work.**

| Questions | MC 1-12 | 1 | 2 | 3 | 4 | 5 | Total |
|---------------|---------|----|----|----|----|----|-------|
| Maximum Marks | 48 | 10 | 10 | 10 | 12 | 10 | 100 |
| Your mark | | | | | | | |

Answers for Multiple Choice Questions:

| # 1 | # 2 | # 3 | # 4 | # 5 | # 6 | #7 | #8 | #9 | # 10 | # 11 | # 12 |
|-----|-----|-----|-----|-----|-----|----|----|----|------|------|------|
| | | | | | | | | | | | |

Part 1. Multiple Choice Questions:

1. Let $u = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. What is the angle between the vectors u and v ?

- (a) $\pi/3$ (b) $-\pi/3$ (c) $2\pi/3$ (d) $-2\pi/3$ (e) $\pi/2$

2. Let A be a 2×2 matrix and $\det A = -3$. What is $\det(2A^2A^T A^{-1})$?

- (a) 36 (b) -18 (c) 18 (d) -36 (e) 9

3. Suppose that a system of equations has the augmented matrix $\left[\begin{array}{cc|c} 1 & h & 3 \\ 0 & -5h - 10 & k - 15 \end{array} \right]$.
If the system is inconsistent, then the values of h and k must be:

- (a) $h = -2, k = 15$ (b) $h = 2, k = 15$ (c) $h \neq -2, k = 15$ (d) $h = -2, k \neq 15$
(e) $h \neq -2, k \neq 15$

4. Given $A^{-1} = \begin{bmatrix} -1 & 3 \\ 4 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, what is the solution of the equation $A\mathbf{x} = \mathbf{b}$?

- (a) $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ (d) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ (e) $\begin{bmatrix} -5 \\ 7 \end{bmatrix}$

5. Let $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ be a diagonal matrix and $P = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$ be an invertible matrix. What is $(PDP^{-1})^{2010}$?

- (a) $\begin{bmatrix} 5 & 4 \\ -6 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & -4 \\ 6 & 5 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Which of the following sets consists of two linearly independent eigenvectors for A ?

- (a) $\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 4+i \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 4-i \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 4+i \\ 1 \end{bmatrix}, \begin{bmatrix} 4-i \\ 1 \end{bmatrix} \right\}$
(d) $\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1-i \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 3+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\}$

7. What is the standard form of the complex number $\frac{2+4i}{1+i}$?

- (a) $1-3i$ (b) $3+i$ (c) $1+2i$ (d) $1-i$ (e) $3-i$

8. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix}$. What is $\det A$?

- (a) 0 (b) 6 (c) 12 (d) 24 (e) 48


9. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \text{ Find } T\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right).$$

- (a) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 11 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} -11 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 9 \\ -6 \end{bmatrix}$

10. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}$. What are the eigenvalues of A ?

- (a) 1, -2, 2 (b) 1, 2, -4 (c) 2, 2, 4 (d) 0, 1, 2 (e) -2, 1, 6

11. Find all values of h such that the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5 \\ -1 \\ h \end{bmatrix}$  are linearly dependent.

- (a) $h = 2$ (b) $h \neq 2$ (c) $h = -2$ (d) $h \neq -2$ (e) $h \neq \pm 2$

12. Which of the following sets are subspaces of \mathbb{R}^3 ?

$$U = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a + b = 1 \right\}, \quad V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a - b = c \right\},$$

$$W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad H = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- (a) U and W (b) V and W (c) V and H (d) U and V (e) V only

Part 2: Long Answer Questions. You must show your work!

1. (10 points) Find the general solution of the following linear system:

$$\begin{cases} 2x_1 + x_2 - x_3 - x_4 = -1 \\ 3x_1 + x_2 + x_3 - 2x_4 = -2 \\ -x_1 - x_2 + 2x_3 + x_4 = 2 \\ -2x_1 - x_2 + 2x_4 = 3 \end{cases}$$

2. (10 points) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

3. (10 points) Use Cramer's rule to find the value of x_3 from the solution to the following linear system:

$$\begin{aligned}x_1 - 2x_2 - 2x_3 &= 7 \\x_2 + 2x_3 &= 4 \\x_1 + x_3 &= 0.\end{aligned}$$

No other method will be accepted.

4. (12 points) Let the matrix A and its reduced row echelon form R be given by the following.

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 & -3 \\ -3 & 1 & -2 & -4 & 0 \\ 4 & -1 & 4 & 7 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} = R.$$

- (a) Find a basis for the column space of A .
- (b) Find a basis for the row space of A .
- (c) Find a basis for the null space of A .
- (d) Verify that the dimension of the column space of A plus the dimension of the null space of A is equal to the number of columns of A .

5. (10 points) Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. You are given that the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 9$.

- (a) Find a basis of the eigenspace corresponding to each eigenvalue.
- (b) Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$; if not, explain why not.