Solutions to Exam

Fall 2010

1. $u=(1,-1,-1,-1)$ $v=(1,1,1,1)$

$\cos\theta = \dfrac{u\cdot v}{\|u\|\,\|v\|}$ $u\cdot v = 1-1-1-1 = -2.$

$\|u\| = \sqrt{4} = 2 = \|v\|.$

$= \dfrac{-2}{4}$ $\theta = 2.0944$

$= -\dfrac{1}{2}$ $\boxed{\theta = 2\pi/3.}$ (c)

2. $\det A = -3$ $\det(2A^2A^TA^{-1})$

$= 2^2 \det(A^2A^TA^{-1})$

$= 4(\det A)^2 \det A \det(A^{-1})$

$= 4(-3)^2(-3)\dfrac{1}{(\det A)}.$

$= 36.$ (a)

3. $\left[\begin{array}{cc|c} 1 & h & 3 \\ 0 & -5h-10 & k-15 \end{array}\right]$ (inconsistent = no solution)

$\boxed{k \neq 15}$ (if $k=15$, then the system is consistent no matter the value of h)

if $-5h-10=0$, then we

(det A).

$= 36.$ (a)

(inconsistent = no solution)

3. $\left[\begin{array}{cc|c} 1 & h & 3 \\ 0 & -5h-10 & k-15 \end{array}\right]$

$\boxed{k \neq 15}$ (if $k = 15$, then the system is consistent no matter the value of h)

$-5h-10=0$ (if $-5h-10=0$, then we have an inconsistent system when $k \neq 15$).

$\boxed{h=-2}$ (d)

(4) $Ax = b$

$A^{-1}Ax = A^{-1}b$

$x = \begin{bmatrix} -1 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -2 & -3 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$

e

(5) $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] R_2 + R_1$

$R_2: -1 \quad -3 \quad 0 \quad 1$
$R_1: \ \ 1 \quad 2 \quad 1 \quad 0$
$\overline{\ \ \ 0 \quad -1 \quad 1 \quad 1}$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] R_1 + 2R_2$

$R_1: 1 \quad 2 \quad 1 \quad 0$
$2R_2: 0 \quad -2 \quad 2 \quad 2$
$\overline{\ \ 1 \quad 0 \quad 3 \quad 2}$

$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right]$ $P^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$

$(PDP^{-1})^{2010} = PD^{2010}P^{-1}$

$= \begin{bmatrix} 1 & 2 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \qquad P^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}.$$

$$(PDP^{-1})^{2010} = PD^{2010}P^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(6) $\det(A - \lambda I)$

$$= \det \begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}$$

$$= (5-\lambda)(3-\lambda) + 2$$

$$= 15 - 8\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 8\lambda + 17$$

$a = 1 \qquad b = -8 \qquad c = 17$

$$\lambda = \frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm \sqrt{-4}}{2}$$

$$= \frac{8 \pm 2i}{2}$$

$$= 4 \pm i$$

$\lambda = 4 + i$

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} R_1 + (i-1)R_2$$

R_1:	$1-i$	-2
$(i-1)R_2$:	$-1+i$	2
	0	0.

$$\begin{bmatrix} 1-i & -2 \\ 0 & 0 \end{bmatrix} \qquad x_2 = t.$$

$(1-i)x_1 = 2t.$

$$x_1 = \frac{2}{(1-i)} t \cdot \frac{(1+i)}{(1+i)}$$

$= 2(1+i)t \qquad (1+i)t$

$$\begin{bmatrix} 1-i & 2 \\ 1 & -1-i \end{bmatrix} R_1 + (i-1)R_2 \qquad \begin{array}{cc} \hline 0 & 0. \end{array}$$

$$\begin{bmatrix} 1-i & -2 \\ 0 & 0 \end{bmatrix} \qquad x_2 = t.$$

$$(1-i)x_1 = 2t.$$

$$x_1 = \frac{2}{(1-i)} t \, \frac{(1+i)}{(1+i)}$$

$$= \frac{2(1+i)t}{1+1} = (1+i)t$$

eigen vector: $((1+i)t, t) = t((1+i), 1)$.

$\lambda = 4 \, i$

$$\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix} (-1-i)R_2 + R_1 \qquad \begin{array}{rcc} R_1: & 1+i & -2 \\ (-1-i)R_2: & -1-i & 2 \\ \hline & 0 & 0. \end{array}$$

$$\begin{bmatrix} 1+i & -2 \\ 0 & 0 \end{bmatrix} \qquad x_2 = r$$

$$(1+i)x_1 = 2r$$

$$x_1 = \frac{2r}{(1+i)}\frac{(1-i)}{(1-i)} = \cancel{2}r(1-i).$$

eigenvector: $r\,((1-i), 1)$

linearly independent eigenvectors: $\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}$

(a)

(7) $$\frac{(2+4i)}{(1+i)}\frac{(1-i)}{(1-i)} = \frac{2-2i+4i+4}{1+1} = \frac{6+2i}{2} = 3+i$$ (b)

(8) $$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix} \quad \begin{matrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{matrix}$$

this column

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix} R_5 - 5R_1$$

$$\begin{array}{rccccc} R_5: & 5 & 5 & 5 & 5 & 6 \\ -5R_1: & -5 & -5 & -5 & -5 & -5 \\ \hline & 0 & 0 & 0 & 0 & 1 \end{array}$$

This elementary row operation doesn't change the determinant

$$= \det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 24. \quad (d)$$

triangular matrix so we can multiply the diagonal entries to get the determinant

(9) $T\left(\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}\right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ $\quad T\left(\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Using the properties of linear transformations we can calculate $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. First, we so we can multiply the diagonal entries to get the determinant

(9) $T\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{bmatrix}5\\-2\end{bmatrix}$ $\quad T\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

Using the properties of linear transformations we can calculate $T\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)$. First, we write $\begin{bmatrix}1\\1\\-2\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$.

$$\begin{bmatrix}1\\1\\-2\end{bmatrix} = 1\begin{bmatrix}0\\1\\-1\end{bmatrix} + 1\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

So $T\begin{pmatrix}1\\1\\-2\end{pmatrix} = T\begin{pmatrix}0\\1\\-1\end{pmatrix} + T\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$= \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}5\\-2\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix} \quad (b)$$

(10) $\det\begin{bmatrix}1-\lambda & 0 & 0\\ 0 & 2-\lambda & 4\\ 0 & 4 & 2-\lambda\end{bmatrix}$

$= (1-\lambda)(2-\lambda)(2-\lambda) - 16(1-\lambda)$

$= (1-\lambda)\left[(2-\lambda)^2 - 16\right]$

$= (1-\lambda)\left[4 - 4\lambda + \lambda^2 - 16\right]$ $\quad P-12 \quad -6, 2$

$= (1-\lambda)\left[\lambda^2 - 4\lambda - 12\right]$ $\quad S-4$

$= (1-\lambda)(\lambda-6)(\lambda+2)$

$\lambda = 1 \quad \lambda = 6 \quad \lambda = -2.$ (e)

(11) $\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 5\\ 0 & 1 & -1\\ -1 & 0 & h\end{bmatrix}$ $\begin{matrix}R_2 - 2R_1\\ \\ R_4 + R_1\end{matrix}$

$$\begin{array}{lrrr} R_4: & -1 & 0 & h\\ R_1: & 1 & 2 & 0\\ \hline & 0 & 2 & h \end{array}$$

$R_2: \ 2 \ -1 \ 5$

$= (1-\lambda)[4-4\lambda+\lambda^2-16]$ $\quad p -12 \quad -6, 2$

$= (1-\lambda)[\lambda^2-4\lambda-12]$ $\quad s -4$

$= (1-\lambda)(\lambda-6)(\lambda+2)$

$\lambda = 1 \quad \lambda = 6 \quad \lambda = -2.$ (e)

(11) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 0 & 1 & -1 \\ -1 & 0 & h \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 + R_1 \end{array}$

$\begin{array}{rrrr} R_4: & -1 & 0 & h \\ R_1: & 1 & 2 & 0 \\ \hline & 0 & 2 & h \end{array}$

$\begin{array}{rrrr} R_2: & 2 & -1 & 5 \\ -2R_1: & -2 & -4 & 0 \\ \hline & 0 & -5 & 5. \end{array}$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & h \end{bmatrix} \begin{array}{l} \\ 5R_2 + R_3 \\ \text{[illegible]} \\ \\ \end{array}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & h \end{bmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\begin{array}{rrrr} R_1: & 1 & 2 & 0 \\ -2R_2: & 0 & -2 & 2 \\ \hline & 1 & 0 & 2 \end{array}$$

(7)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & h \\ 0 & 0 & 0 \end{bmatrix} R_3 - 2R_2$$

$$\begin{array}{rrrr} R_3: & 0 & 2 & h \\ -2R_2: & 0 & -2 & 2 \\ \hline & 0 & 0 & h+2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & h+2 \\ 0 & 0 & 0 \end{bmatrix}$$

If $h=-2$, then our matrix is in row-reduced echelon form and the third column isn't an elementary vector. (so linearly dependent).

If $h \neq -2$, then we can further reduce it and get 3 distinct elementary vectors (Linearly independent).

So, $h=-2$ (c)

(12) $\underline{\underline{U}}$ Let $(a, b, 0)$ be in U (so $a+b=1$). and let c be a scalar. Consider $(ca, cb, 0)$.

So $ca+cb = c(a+b) = c(1) = c \neq 1$

Thus not a subspace.

If $h \neq -2$, then we can further reduce it and get 3 distinct elementary vectors (linearly independent).

So, $h = -2$ (c)

(12) <u>U</u> Let $(a, b, 0)$ be in U (so $a+b=1$) and let c be a scalar. Consider $(ca, cb, 0)$.

So $ca + cb = c(a+b) = c(1) = c. \neq 1$

Thus not a subspace.

<u>V</u> Let (a, b, c) be in V (so $a-b=c$). Let d be a scalar. Consider (da, db, dc).

So $da - db = d(a-b) = dc$ ✓ First requirement holds

Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be in V. So $a_1 - b_1 = c_1$ and $a_2 - b_2 = c_2$. Consider

$(a_1+a_2, b_1+b_2, c_1+c_2)$. So $a_1+a_2-(b_1+b_2)$ 8

$= (a_1-b_1)+(a_2-b_2)$

$= c_1 + c_2$ ✓

Thus V is a subspace.

~~(a)~~ (b) (c) ~~(d)~~ (e)

W is not a subspace since $2\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not in W

~~(a)~~ ~~(b)~~ (c) ~~(d)~~ (e)

H is a subspace

$$c\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

C

$$\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Part 2

(1) $\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & -1 \\ 3 & 1 & 1 & -2 & -2 \\ -1 & -1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 2 & 3 \end{array}\right]$ $R_3 \leftrightarrow R_1$

$\left[\begin{array}{cccc|c} -1 & -1 & 2 & 1 & 2 \end{array}\right.$ R_2+3R_1

R_2: 3 1 1 -2 -2

$3R_1$: -3 -3 +6 3 6

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Part 2

(1) $$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & -1 \\ 3 & 1 & 1 & -2 & -2 \\ -1 & -1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 2 & 3 \end{array}\right] R_3 \leftrightarrow R_1$$

$$\left[\begin{array}{cccc|c} -1 & -1 & 2 & 1 & 2 \\ 3 & 1 & 1 & -2 & -2 \\ 2 & 1 & -1 & -1 & -1 \\ -2 & -1 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} \\ R_2+3R_1 \\ \\ R_4+R_3 \end{array}$$

$$\begin{array}{rrrrrr} R_2: & 3 & 1 & 1 & -2 & -2 \\ 3R_1: & -3 & -3 & +6 & 3 & 6 \\ \hline & 0 & -2 & 7 & 1 & 4 \end{array}$$

$$\begin{array}{rrrrrr} R_4: & -2 & -1 & 0 & 2 & 3 \\ R_3: & 2 & 1 & -1 & -1 & -1 \\ \hline & 0 & 0 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{cccc|c} -1 & -1 & 2 & 1 & 2 \\ 0 & -2 & 7 & 1 & 4 \\ 2 & 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \\ R_3+2R_1 \\ \\ \end{array}$$

$$\begin{array}{rrrrrr} R_3: & 2 & 1 & -1 & -1 & -1 \\ 2R_1: & -2 & -2 & 4 & 2 & 4 \\ \hline & 0 & -1 & 3 & 1 & 3 \end{array}$$

$$\left[\begin{array}{cccc|c} -1 & -1 & 2 & 1 & 2 \\ 0 & -2 & 7 & 1 & 4 \\ 0 & -1 & 3 & 1 & 3 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} R_1(-1) \\ R_2 \leftrightarrow R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & -1 & -2 \\ 0 & -1 & 3 & 1 & 3 \\ 0 & -2 & 7 & 1 & 4 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right] R_3 - 2R_2$$

$$\begin{array}{rrrrrr} R_3: & 0 & -2 & 7 & 1 & 4 \\ -R_2: & 0 & 2 & -6 & -2 & -6 \\ \hline & 0 & 0 & 1 & -1 & -2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & -1 & -2 \\ 0 & -1 & 3 & 1 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ \\ R_4 + R_3 \end{array}$$

$$\begin{array}{rrrrrr} R_1: & 1 & 1 & -2 & -1 & -2 \\ R_2: & 0 & -1 & 3 & 1 & 3 \\ \hline & 1 & 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -3 & -1 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 + 3R_3 \end{array}$$

$$\begin{array}{rrrrrr} R_1: & 1 & 0 & 1 & 0 & 1 \\ -R_3: & 0 & 0 & -1 & 1 & 2 \\ \hline & 1 & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{array}{rrrrrr} R_2: & 0 & 1 & -3 & -1 & -3 \\ 3R_3: & 0 & 0 & 3 & -3 & -6 \\ \hline & 0 & 1 & 0 & -4 & -9 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = t.$

$x_3 = t - 2$

$x_2 = +4t - 9$

$x_1 = 3 - t.$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\begin{array}{rl} R_2: & 0 \quad 1 \quad -3 \quad -1 \quad -3 \\ 3R_3: & 0 \quad 0 \quad 3 \quad -3 \quad -6 \\ \hline & 0 \quad 1 \quad 0 \quad -4 \quad -9 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = t.$

$x_3 = t - 2$

$x_2 = +4t - 9$

$x_1 = 3 - t.$

Solution: $(3-t, 4t-9, t-2, t)$

(2) $$\left[\begin{array}{ccc:ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] R_3 - R_1$$

$$\begin{array}{rl} R_3: & 1 \quad 0 \quad 3 \quad 0 \quad 0 \quad 1 \\ -R_1: & -1 \quad -1 \quad 2 \quad -1 \quad 0 \quad 0 \\ \hline & 0 \quad -1 \quad 5 \quad -1 \quad 0 \quad 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 5 & -1 & 0 & 1 \end{array}\right] R_3 + R_2$$

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$$\begin{array}{rcccccc} R_3: & 0 & -1 & 5 & -1 & 0 & 1 \\ R_2: & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline & 0 & 0 & 6 & -1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right] R_1 - R_2$$

$$\begin{array}{rcccccc} R_1: & 1 & 1 & -2 & 1 & 0 & 0 \\ -R_2: & 0 & -1 & -1 & 0 & -1 & 0 \\ \hline & 1 & 0 & -3 & 1 & -1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right] 2R_1 + R_3$$

$$\begin{array}{rcccccc} 2R_1: & 2 & 0 & -6 & 2 & -2 & 0 \\ R_3: & 0 & 0 & 6 & -1 & 1 & 1 \\ \hline & 2 & 0 & 0 & 1 & -1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right] R_3 - 6R_2$$

$$\begin{array}{rcccccc} R_3: & 0 & 0 & 6 & -1 & 1 & 1 \\ -6R_2: & 0 & -6 & -6 & 0 & -6 & 0 \\ \hline & 0 & -6 & 0 & -1 & -5 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & -6 & 0 & -1 & \cancel{4}-5 & 1 \\ 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1/2 \\ R_2/-6 \\ R_3/6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/6 & 5/6 & -1/6 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/6 & 5/6 & -1/6 \\ -1/6 & 1/6 & 1/6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right] R_3/6$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/6 & 5/6 & -1/6 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/6 & 5/6 & -1/6 \\ -1/6 & 1/6 & 1/6 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix}$

$\det A = 1 - 4 + 2 = -1$

$\det A_3 = -8 - 7 = -15$

$$x_3 = \frac{\det A_3}{\det A} = \frac{-15}{-1} = 15$$

(4a) $\{(1,-3,4), (-2,1,-1), (2,-2,4)\}$

(b) $$R^T = \begin{array}{cl} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ R_4 - R_1 \\ R_5 - R_1 \end{array} \end{array}$$

$$\begin{array}{rrrr} R_4: & 1 & 1 & 1 \\ -R_1: & -1 & 0 & 0 \\ \hline & 0 & 1 & 1 \end{array}$$

$$\begin{array}{rrrr} R_5: & 1 & 1 & -1 \\ -R_1: & -1 & 0 & 0 \\ \hline & 0 & 1 & -1 \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ R_4 - R_2 \\ R_5 - R_2 \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ R_4 - R_3 \\ R_5 + R_3 \end{array} \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

... necessary to put R^T in

$$\begin{array}{c}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\end{array}\begin{array}{l} \\ R_4 - R_2 \\ R_5 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{array}{l} \\ \\ \\ R_4 - R_3 \\ R_5 + R_3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

not necessary to put R^T in row reduced echelon form for this example, since the dimension of the column space is the same as the dimension of the row space

Thus the dimension of row space is 3 and so the basis has to be $\longrightarrow$

$\{(1,-2,2,1,-3), (-3,1,-2,-4,0), (4,-1,4,7,-1)\}$ (2)

(c) Find the solution of homogeneous system in order to find the basis of the null space

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$x_4 = t$
$x_5 = r.$

$x_1 + t + r = 0$
$x_1 = -t - r$

$x_2 = -t - r$
$x_3 = -t + r.$

Solution: $(-t-r, -t-r, -t+r, t, r)$

$= (-t, -t, -t, t, 0) + (-r, -r, r, 0, r)$

$= t(-1, -1, -1, 1, 0) + r(-1, -1, 1, 0, 1)$

Basis of nullspace: $\{(-1,-1,-1,1,0), (-1,-1,1,0,1)\}$

(d) dim col space + dim null space = # columns

$3 + 2 = 5$ ✓

(5a) ~~[illegible]~~ $A - \lambda I = \begin{bmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{bmatrix}$

Basis of nullspace: $\{(-1,-1,-1,1,0),\ (-1,-1,1,0,1)\}$

(d) dim col space + dim null space = # columns

$$3 + 2 = 5 \checkmark$$

(5a) ~~[illegible]~~ $A - \lambda I = \begin{bmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{bmatrix}$

$\lambda_1 = 2$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_3 = t \\ x_2 = r \end{array}$$

$$2x_1 - r + 6t = 0.$$

$$2x_1 = r - 6t$$

$$x_1 = \frac{r}{2} - 3t.$$

Solution: $\left(\frac{r}{2} - 3t, r, t\right)$

$$= \left(\frac{r}{2}, r, 0\right) + (-3t, 0, t)$$

$$= r(1/2, 1, 0) + t(-3, 0, 1)$$

Basis of eigenspace: $\{(1/2, 1, 0), (-3, 0, 1)\}$

$\lambda_3 = 9$

$$\begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} R_3 - R_2 \qquad \begin{array}{rrrr} R_3: & 2 & -1 & -1 \\ -R_2: & -2 & 8 & -6 \\ \hline & 0 & 7 & -7 \end{array}$$

$$\begin{bmatrix} -5 & -1 & 6 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{bmatrix} R_1 + 5R_2 \qquad \begin{array}{rrrr} R_1: & -5 & -1 & 6 \\ 5R_2: & 5 & -20 & 15 \\ \hline & 0 & -21 & 21 \end{array}$$

$$\begin{bmatrix} -5 & -1 & 6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} R_3 - R_2$$

$[-5 \quad -1 \quad 6] \; R_1 + R_2 \qquad R_1: -5 \quad -1 \quad 6$

$$\begin{bmatrix} -5 & -1 & 6 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{bmatrix} R_1 + 5R_2 \qquad \begin{array}{rrrr} R_1: & -5 & -1 & 6 \\ 5R_2: & 5 & -20 & 15 \\ \hline & 0 & -21 & 21 \end{array}$$

$$\begin{bmatrix} -5 & -1 & 6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} -5 & -1 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_1 + R_2 \qquad \begin{array}{rrrr} R_1: & -5 & -1 & 6 \\ R_2: & 0 & 1 & -1 \\ \hline & -5 & 0 & 5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_3 = t$$

$x_1 = t$ Solution: $t(1,1,1)$
$x_2 = t$

Basis for eigenspace: $\{(1,1,1)\}$

(b) It is diagonalizable since we have 3 vectors in the combined eigenspace of $\lambda_1, \lambda_2, \lambda_3$.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad P = \begin{bmatrix} 1/2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$