MATH 3705A
Test 3 Solutions
March 11, 2014

[Marks] Questions 1-3 are multiple choice. Circle the correct answer. Only the answer will be marked.

[5] 1. Let $f(x) = x$ for $0 \le x < 2$ and $f(x+2) = f(x)$ for all x. The Fourier series of f is $\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\Big]$, with

(a) $a_0 = 2$, $a_n = 0$ and $b_n = -\frac{2}{n\pi}$ for $n \ge 1$

(b) $a_0 = 4$, $a_n = 0$ and $b_n = -\frac{4}{n\pi}$ for $n \ge 1$

(c) $a_0 = 2$, $a_n = 0$ and $b_n = \frac{2}{n\pi}$ for $n \ge 1$

(d) $a_0 = 0$, $a_n = 0$ and $b_n = -\frac{4}{n\pi}$ for $n \ge 1$

(e) None of the above

Answer: (a)

[5] 2. Let $f(x) = \left\{ \begin{array}{l} 2,\ 0 \le x \le 2 \\ 4,\ 2 < x \le 3 \end{array} \right\}$. At $x = 10$, the Fourier sine series of f on $[0,3]$ converges to

(a) 2 (b) -2 (c) -3 (d) -4 (e) None of these

Answer: (c)

[5] 3. Let $f(x)$ be as in Question 2. At $x = 10$, the Fourier cosine series of f on $[0,3]$ converges to

(a) 0 (b) 2 (c) 3 (d) 4 (e) None of these

Answer: (c)

[5] 4. Find the Fourier cosine series of $f(x) = x$ on $[0,1]$.

Solution:

The cosine series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$, with $a_0 = 2\int_0^1 x\,dx = 1$, and

$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{n\pi} x \sin(n\pi x)\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx = \frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1$

$= \frac{2}{n^2\pi^2}\Big[(-1)^n - 1\Big]$, $n \ge 1$. The cosine series is $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\Big[(-1)^n - 1\Big]\cos(n\pi x)$.

[10] 5. The solution of the heat equation $w_{xx} = \frac{1}{\alpha^2} w_t$, $0 < x < L$, $t > 0$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, $t > 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \frac{1}{9}u_t$, $0 < x < 3$, $t > 0$, which satisfies the boundary conditions $u(0,t) = 2$, $u(3,t) = 8$, $t > 0$, and the initial condition $u(x,0) = 4x + 2$, $0 \leq x \leq 2$. Write down the complete solution $u(x,t)$.

Solution:

$u(x,t) = v(x) + w(x,t)$, $v(x) = \dfrac{8-2}{3}x + 2 = 2x + 2$,

$w(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t}$, and $w(x,0) = u(x,0) - v(x) = 2x \;\Rightarrow$

$$b_n = \frac{2}{3}\int_0^3 2x \sin\left(\frac{n\pi x}{3}\right) dx = -\frac{4}{n\pi} x \cos\left(\frac{n\pi x}{3}\right)\Big|_0^3 + \frac{4}{n\pi}\int_0^3 \cos\left(\frac{n\pi x}{3}\right) dx = \frac{-12(-1)^n}{n\pi}$$

$$\Rightarrow\; u(x,t) = 2x + 2 + \sum_{n=1}^{\infty} \frac{-12(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{3}\right) e^{-n^2\pi^2 t}.$$