Circle: DGD 1 (Andrew) LAST NAME (in capitals): Solutions

DGD 2 (Elizabeth) First name:

Marks: /11 Student number:

MAT 1348A (Prof. M. Šajna) — Eighth Homework Assignment
Due March 25, 2015 by 4:00pm

Instructions: Print out this document and staple the pages. You may write on both sides of the paper or insert additional pages if necessary.

Submit a finished, presentable product. *Drafts and illegible papers will not be marked.* Show all relevant work to receive full credit.

Submit the assignment to your TA in the DGD or in the *submission box labeled MAT 1348A* in the Department of Mathematics and Statistics. Late assignments will not be accepted.

1. At random, you choose a subset S of $\mathbb{Z} \times \mathbb{Z}$. What is the smallest cardinality of S that guarantees that your set S contains two elements (m, n) and (m', n') such that $m \equiv m'$ (mod 6) and $n \equiv n'$ (mod 9)? [3pts]

Boxes: pairs (x, y), where x is a remainder mod 6 and y is a remainder mod 9:

objects: pairs $(m, n) \in S$

(pair (m, n) is placed in box labelled (x, y) if $m \equiv x \pmod 6$ and $n \equiv y \pmod 9$)

#boxes: $6 \cdot 9 = 54$

By the Pigeonhole Principle, we need 55 pairs to guarantee that at least 1 box contains at least 2 objects (clearly, 54 does not suffice).

Answer: <u>55</u>

2. How many integers in the set $\{100, 101, \ldots, 1000\}$ are divisible by 4 or 7? (This is **inclusive or**.) How many are divisible by neither? [4pts]

Let $A = \{ n \in \mathbb{Z} : 100 \leq n \leq 1000,\ 4 | n \}$

$B = \{ n \in \mathbb{Z} : 100 \leq n \leq 1000,\ 7 | n \}$

We are looking for $|A \cup B|$ and $|\overline{A \cup B}|$,

where $\mathcal{U} = \{100, 101, \ldots, 1000\}$.

By PIE:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Now

$$|A| = \left\lfloor \frac{1000}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor = 250 - 24 = 226$$

$$|B| = \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{99}{7} \right\rfloor = 142 - 14 = 128$$

Since $A \cap B = \{ n \in \mathbb{Z} : 100 \leq n \leq 1000,\ 28 | n \}$,

we have $|A \cap B| = \left\lfloor \frac{1000}{28} \right\rfloor - \left\lfloor \frac{99}{28} \right\rfloor = 35 - 3 = 32$

Hence $|A \cup B| = 226 + 128 - 32 = 322$

and $|\overline{A \cup B}| = |\mathcal{U}| - |A \cup B| = 1000 - 99 - 322 = 579$

Answer: 322 elements of $\{100, \ldots, 1000\}$ are divisible by 4 or 7, and 579 by neither.

3. A student wishes to arrange on a shelf, in a row, 5 out of her 12 textbooks (which includes a copy of Rosen's Discrete Mathematics and a copy of Stewart's Calculus). How many ways can this be done if [4pts]

(a) at least one of Rosen and Stewart must be on the shelf?

(b) exactly one of Rosen and Stewart must be on the shelf?

(a) # ways do arrange 5 out of 12 books so that <u>neither</u> Rosen nor Stewart is among the 5 : $P(10,5)$

ways do arrange 5 out of 12 books without restriction : $P(12,5)$

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ways do arrange 5 out of 12 books so that at least one of R and S is included :

$$P(12,5) - P(10,5) =$$

$$= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 - 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 =$$

$$= 95040 - 30240 = \underline{\underline{64\,800}}$$

(b) # ways do choose 4 out of 10 books (excluding R and St.) : $\binom{10}{4}$

ways do choose R or St. : 2

ways do arrange the 5 chosen books : $5!$

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ways do arrange 5 out of 12 books, including exactly one of R and St. (Product Rule):

$$\binom{10}{4} \cdot 2 \cdot 5! = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = \underline{\underline{50\,400}}$$