Concordia University

Department of Computer Science & Software Engineering

COMP 232/4 **Mathematics for Computer Science** Winter 2015

Assignment 1 Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

   (a) $(\underbrace{(p \vee r)}_{\mathbf{a}} \wedge \underbrace{(q \vee r)}_{\mathbf{b}}) \leftrightarrow \underbrace{((p \wedge q) \vee r)}_{\mathbf{c}}$

   Solution: Tautology.

p	q	r	$\overbrace{p \vee r}^{\mathbf{a}}$	$\overbrace{q \vee r}^{\mathbf{b}}$	$\mathbf{a} \wedge \mathbf{b}$	$p \wedge q$	$\overbrace{(p \wedge q) \vee r}^{\mathbf{c}}$	$(\mathbf{a} \wedge \mathbf{b}) \leftrightarrow \mathbf{c}$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	T

   (b) $\underbrace{(p \wedge \underbrace{(\neg q \rightarrow \neg p)}_{\mathbf{a}})}_{\mathbf{b}} \rightarrow q$

   Solution: Tautology.

p	q	$\neg p$	$\neg q$	$\overbrace{\neg q \rightarrow \neg p}^{\mathbf{a}}$	$\overbrace{p \wedge \mathbf{a}}^{\mathbf{b}}$	$\mathbf{b} \rightarrow q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

   (a) $(p \to r) \land (q \to r) \equiv ((p \land q) \to r)$

   Solution: Invalid.
   If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

   (b) $((p \lor q) \land (\neg p \lor r)) \equiv (q \lor r)$

   Solution: Invalid.
   If $p = T$, $q = T$, and $r = F$ then the LHS is False, while the RHS is True.

3. Five persons, anonymously known as P_1, P_2, P_3, P_4, P_5, are suspected of being involved in a crime. Suppose we have the following information:

   (a) If P_5 is involved then so is P_3.

   (b) If P_2 is involved then so are P_5 and P_1.

   (c) Either P_1 or P_2, or both, are involved.

   (d) Either P_3 or P_4, but not both, are involved.

   (e) P_4 and P_1 are either both involved or neither is.

   Determine which persons were involved in the crime. Explain your reasoning.

   Solution:

   We write the statements in propositional logic, using p_i to denote that P_i is involved, and $\neg p_i$ to denote that P_i is not involved,

   (a) $p_5 \to p_3$

   (b) $p_2 \to (p_5 \wedge p_1)$

   (c) $p_1 \vee p_2$

   (d) $p_3 \oplus p_4$

   (e) $p_4 \leftrightarrow p_1$

   From (d) we have two possibilities, namely, $p_3 = T$, $p_4 = F$, or $p_3 = F$, $p_4 = T$. We consider each of these possibilities separately.

   i If $p_3 = T$ and $p_4 = F$ then it follows from (e) that $p_1 = F$. Since $p_1 = F$ it follows from (c) that $p_2 = T$. Since $p_2 = T$ it follows in particular from (b) that $p_1 = T$. So p_1 is both True and False, which is a contradiction. Thus the assumption that $p_3 = T$, $p_4 = F$ cannot be true, in other words, the assumption that $p_3 = T$, $p_4 = F$ does not lead to a solution of the crime problem.

   ii If $p_3 = F$ and $p_4 = T$ then it follows from (a) that $p_5 = F$. Since $p_5 = F$ it follows from (b) that $p_2 = F$. Since $p_2 = F$ it follows from (c) that $p_1 = T$. Finally, since both $p_4 = T$ and $p_1 = T$, we see that (e) is also satisfied.

   Having considered the two cases above, we see that there is only one solution, namely p_1 and p_4 were involved, but p_2, p_3, and p_5 are not.

4. Write the following statements in predicate form, using logical operators $\wedge$, $\vee$, $\neg$, and quantifiers $\forall$, $\exists$. Below $\mathbb{Z}^+$ denotes all positive integers $\{1, 2, 3, \ldots\}$.

   (a) For any $x, y \in \mathbb{Z}^+$ the equation $x^2 + y^2 - z = 0$ has a solution $z \in \mathbb{Z}^+$.

   Solution:

   $$\forall x, y \in \mathbb{Z}^+ \, \exists z \in \mathbb{Z}^+ \big(x^2 + y^2 - z = 0\big)$$

   (b) The equation $x^3 + y^3 = z^3$ has no solutions $x, y, z \in \mathbb{Z}^+$.

   Solution:

   $$\neg\Big(\exists\, x, y, z \in \mathbb{Z}^+ \big(x^3 + y^3 = z^3\big)\Big)$$

   or equivalently

   $$\forall\, x, y, z \in \mathbb{Z}^+ \big(x^3 + y^3 \neq z^3\big)$$

   (c) The difference between two positive integers can be arbitrarily large.

   Solution:

   $$\forall z \in \mathbb{Z}^+ \, \exists x, y \in \mathbb{Z}^+ \Big(x - y > z\Big)$$

5. Let P and Q be predicates on the set S, where S has two elements, say, $S = \{a, b\}$. Then the statement $\forall x P(x)$ can also be written in full detail as $P(a) \wedge P(b)$. Rewrite each of the statements below in a similar fashion, using P, Q, and logical operators, but without using quantifiers.

   (a) $\forall x \forall y (P(x) \vee Q(y))$

   Solution:

$$\begin{aligned} & \forall x \Big(P(x) \vee \big(Q(a) \wedge Q(b) \big) \Big) \\ \equiv\ & \big(P(a) \wedge P(b) \big) \vee \big(Q(a) \wedge Q(b) \big) \\ \equiv\ & \big(P(a) \vee Q(a) \big) \wedge \big(P(a) \vee Q(b) \big) \wedge \big(P(b) \vee Q(a) \big) \wedge \big(P(b) \vee Q(b) \big) \end{aligned}$$

   (b) $\exists x P(x) \wedge \exists x Q(x)$

   Solution:

$$\big(P(a) \vee P(b) \big) \wedge \big(Q(a) \vee Q(b) \big)$$

   (c) $\exists x \exists y (P(x) \wedge Q(y))$

   Solution:

$$\begin{aligned} & \exists y \Big(\big(P(a) \vee P(b) \big) \wedge Q(y) \Big) \\ \equiv\ & \big(P(a) \vee P(b) \big) \wedge \big(Q(a) \vee Q(b) \big) \\ \equiv\ & \big(P(a) \wedge Q(a) \big) \vee \big(P(a) \wedge Q(b) \big) \vee \big(P(b) \wedge Q(a) \big) \vee \big(P(b) \wedge Q(b) \big) \end{aligned}$$

   (d) $\forall x \exists y (P(x) \wedge Q(y))$

   Solution:

$$\begin{aligned} & \forall x \Big(P(x) \wedge \big(Q(a) \vee Q(b) \big) \Big) \\ \equiv\ & \Big(P(a) \wedge \big(Q(a) \vee Q(b) \big) \Big) \wedge \Big(P(b) \wedge \big(Q(a) \vee Q(b) \big) \Big) \\ \equiv\ & \Big(\big(P(a) \wedge Q(a) \big) \vee \big(P(a) \wedge Q(b) \big) \Big) \wedge \Big(\big(P(b) \wedge Q(a) \big) \vee \big(P(b) \wedge Q(b) \big) \Big) \end{aligned}$$

6. Let the Universe of Discourse for x be the set of all students in this class and for y be the set of all countries in the world. Let $P(x, y)$ denote student x has visited country y and $Q(x, y)$ denote student x has a friend in country y. Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

(a) Carlos has visited Bulgaria.

Solution:

$$P(\mathit{Carlos}, \mathit{Bulgaria})$$

(b) Every student in this class has visited the United States.

Solution:

$$\forall x\ P(x, \mathit{US})$$

(c) Every student in this class has visited some country in the world.

Solution:

$$\forall x \exists y\ P(x, y)$$

(d) There is no country that every student in this class has visited.

Solution:

$$\forall y \exists x\ \big(\neg P(x, y)\big)$$

(e) There are two students in this class, who between them, have a friend in every country in the world.

Solution:

$$\exists x_1 \exists x_2\ \Big(x_1 \neq x_2 \wedge \forall y\ \big(Q(x_1, y) \vee Q(x_2, y)\big)\Big)$$

7. Determine the truth value of each of the following statements if the universe of discourse of each variable consists of all real numbers.

   (a) $\forall x \exists y (x + y = 1)$

   Solution: True (let $y = 1 - x$).

   (b) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$

   Solution: False.

   (c) $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$

   Solution: False (let $x = 0$).

   (d) $\forall x \forall y \exists z (z = (x + y)/2)$

   Solution: True.

8. Negate the following statements and transform the negation so that negation symbols immediately precede predicates.

   (a) $\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)$

   Solution:

$$\begin{aligned}
& \neg\Big(\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)\Big) \\
\equiv\ & \neg\Big(\exists x \exists y P(x,y)\Big) \vee \neg\Big(\forall x \forall y Q(x,y)\Big) \\
\equiv\ & \Big(\forall x \forall y \neg P(x,y)\Big) \vee \Big(\exists x \exists y \neg Q(x,y)\Big)
\end{aligned}$$

   (b) $\exists x \exists y (Q(x,y) \leftrightarrow Q(y,x))$

   Solution:

$$\begin{aligned}
& \neg\Big(\exists x \exists y \big(Q(x,y) \leftrightarrow Q(y,x)\big)\Big) \\
\equiv\ & \forall x \forall y \Big(\neg\big(Q(x,y) \leftrightarrow Q(y,x)\big)\Big) \\
\equiv\ & \forall x \forall y \neg\Big(\big(Q(x,y) \rightarrow Q(y,x)\big) \wedge \big(Q(y,x) \rightarrow Q(x,y)\big)\Big) \\
\equiv\ & \forall x \forall y \Big(\neg\big(Q(x,y) \rightarrow Q(y,x)\big) \vee \neg\big(Q(y,x) \rightarrow Q(x,y)\big)\Big) \\
\equiv\ & \forall x \forall y \Big(\neg\big(\neg Q(x,y) \vee Q(y,x)\big) \vee \neg\big(\neg Q(y,x) \vee Q(x,y)\big)\Big) \\
\equiv\ & \forall x \forall y \Big(\big(Q(x,y) \wedge \neg Q(y,x)\big) \vee \big(Q(y,x) \wedge \neg Q(x,y)\big)\Big)
\end{aligned}$$

   (c) $\forall y \exists x \exists z (T(x,y,z) \vee Q(x,y))$

   Solution:

$$\begin{aligned}
& \neg\Big(\forall y \exists x \exists z \big(T(x,y,z) \vee Q(x,y)\big)\Big) \\
\equiv\ & \exists y \forall x \forall z \neg\big(T(x,y,z) \vee Q(x,y)\big) \\
\equiv\ & \exists y \forall x \forall z \big(\neg T(x,y,z) \wedge \neg Q(x,y)\big)
\end{aligned}$$