Seventh Edition

CHAPTER

4

MECHANICS OF MATERIALS

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Pure Bending

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Pure Bending

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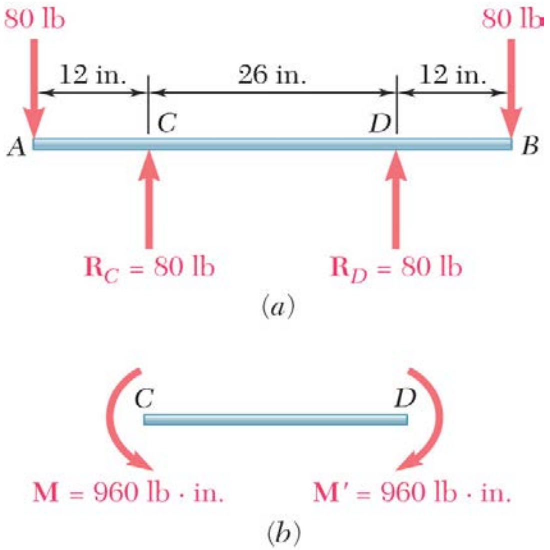


Pure Bending:

Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Fig. 4.2 (a) Free-body diagram of the barbell pictured in the chapter opening photo and (b) Free-body diagram of the center bar portion showing pure bending.

Other Loading Types

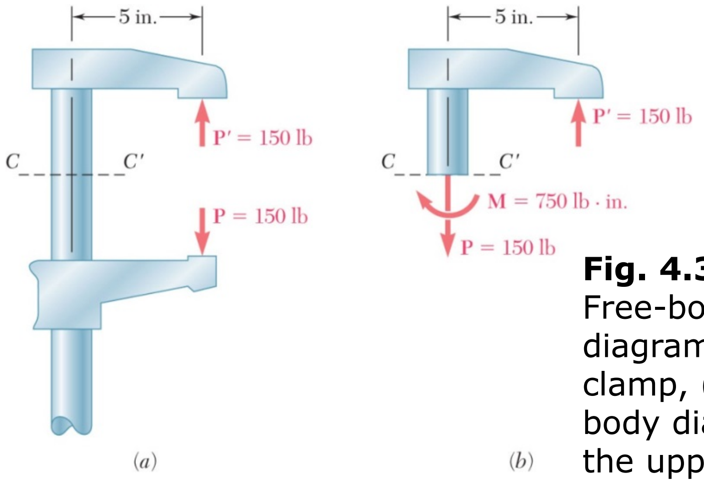


Fig. 4.3 (a) Free-body diagram of a clamp, (b) free-body diagram of the upper portion of the clamp.

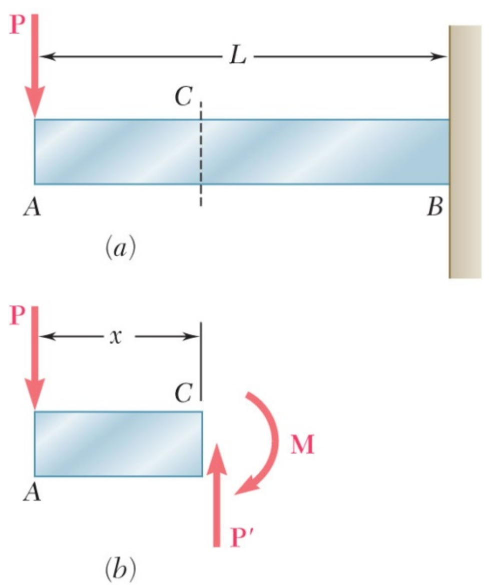


Fig. 4.4 (a) Cantilevered beam with end loading. (b) As portion *AC* shows, beam is not in pure bending.

- *Eccentric Loading*: Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple
- *Transverse Loading*: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple
- *Principle of Superposition*: The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.

Symmetric Member in Pure Bending

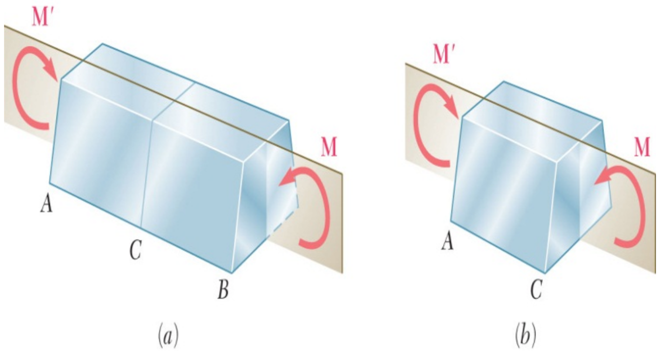


Fig. 4.5 (a) A member in a state of pure bending. (b) Any intermediate portion of *AB* will also be in pure bending.

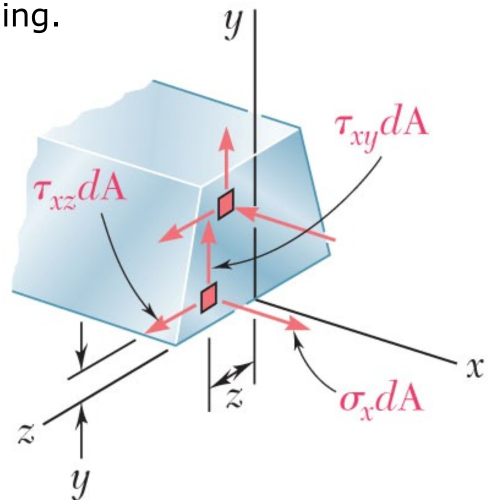


Fig. 4.6 Summation of the infinitesimal stress elements must produce the equivalent pure-bending moment.

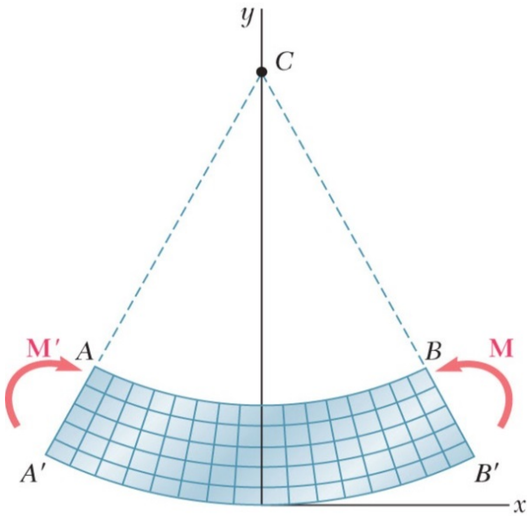
- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple **M** consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about *any* axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x \, dA = 0$$

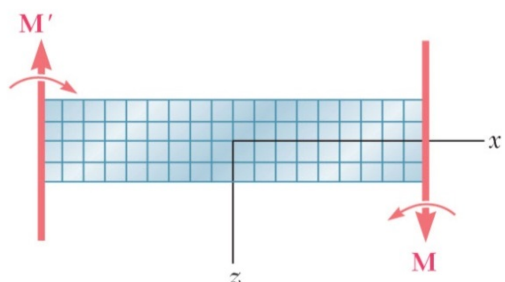
$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$

Bending Deformations



(*a*) Longitudinal, vertical section (plane of symmetry)



(*b*) Longitudinal, horizontal section

Fig. 4.9 Member subject to pure bending shown in two views. (a) Longitudinal, vertical view (plane of symmetry) and (b) Longitudinal, horizontal view.

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending

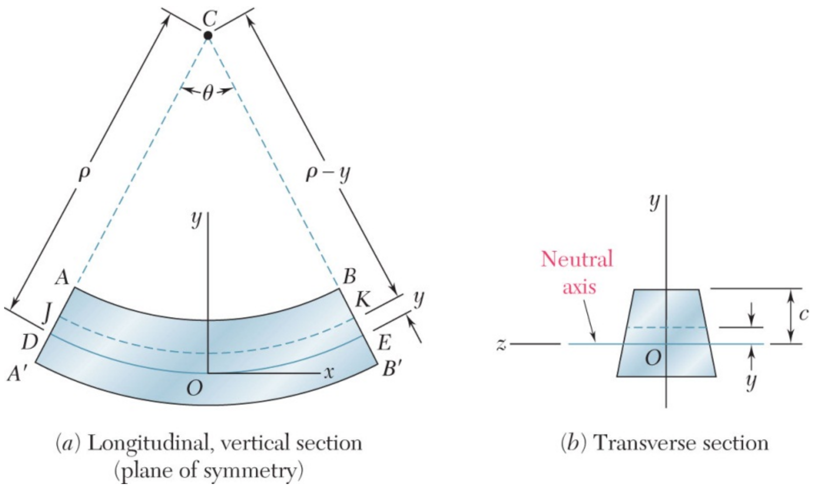


Fig. 4.10 Kinematic definitions for pure bending. (a) Longitudinal-vertical view and (b) Transverse section at origin.

Consider a beam segment of length L.

After deformation, the length of the neutral surface remains L. At other sections,

$$L' = (\rho - y)\theta$$

$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Stress Due to Bending

- For a linearly elastic and homogeneous material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral axis is zero. Therefore, the neutral axis must pass through the section centroid.

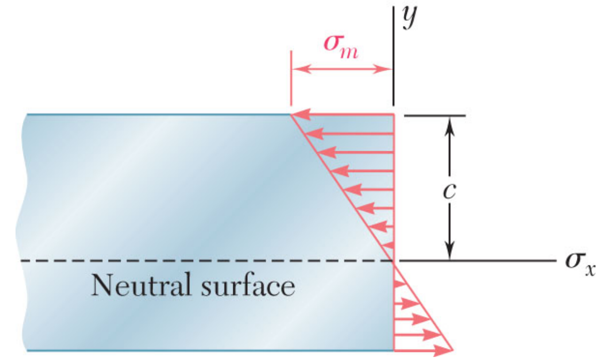


Fig. 4.11 Bending stresses vary linearly with distance from the neutral axis.

- For static equilibrium,

$$M = \int(-y\sigma_x \, dA) = \int(-y)\left(-\frac{y}{c}\sigma_m\right) dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$

Beam Section Properties

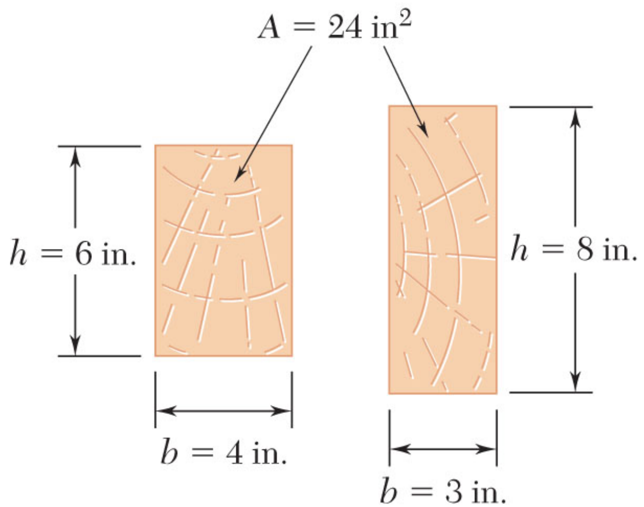


Fig. 4.12 Wood beam cross sections.

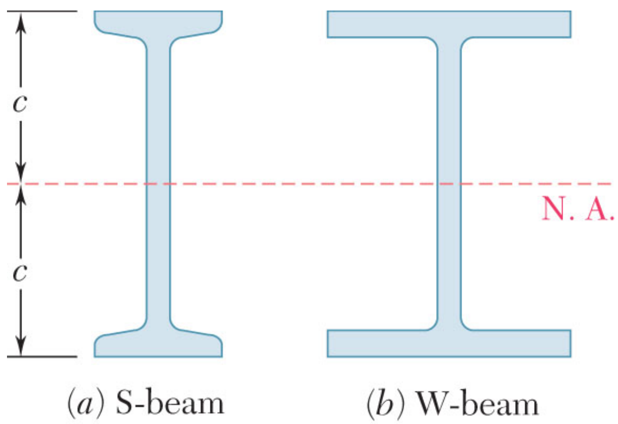


Fig. 4.13 Two type of steel beam cross sections. (a) S-beam and (b) W-beam

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

I = section moment of inertia

$$S = \frac{I}{c} = \text{section modulus}$$

A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^3 = \frac{1}{6}Ah$$

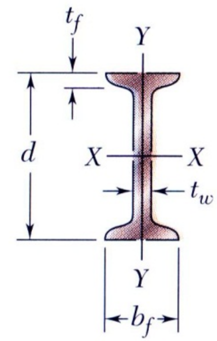
Between two beams with the same cross sectional area, the beam with the larger depth h will be more effective in resisting bending.

- Structural steel beams are designed to have a large section modulus.

Properties of American Standard Shapes

Appendix C. **Properties of Rolled-Steel Shapes (SI Units)**

S Shapes
(American Standard Shapes)



Designation†	Area A, mm²	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm⁴	Axis X-X S_x 10^3 mm³	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm⁴	Axis Y-Y S_y 10^3 mm³	Axis Y-Y r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

Deformations in a Transverse Cross Section

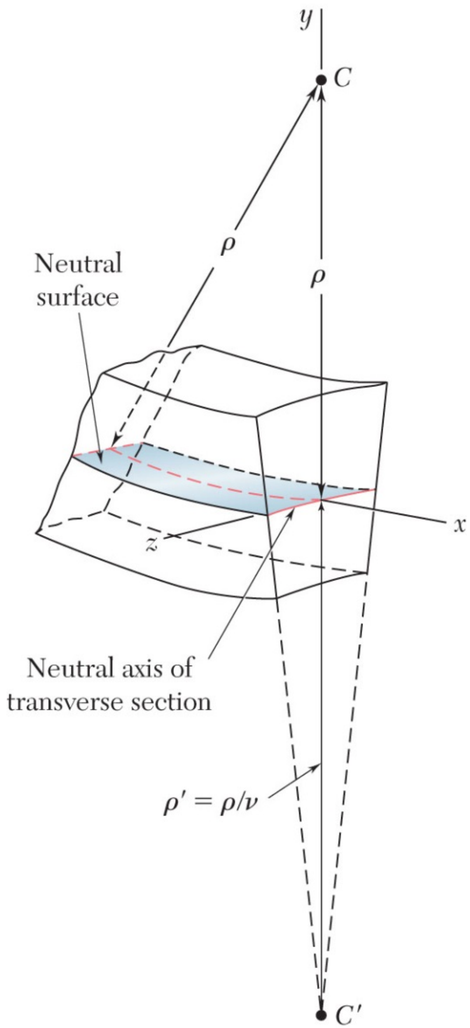


Fig. 4.16 Deformation of a transverse cross section.

- Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$
$$= \frac{M}{EI}$$

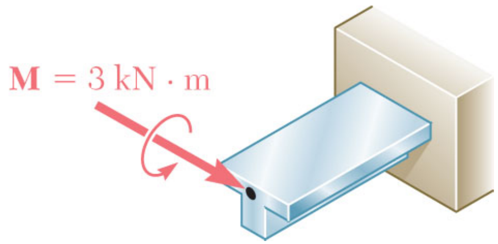
- Although transverse cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

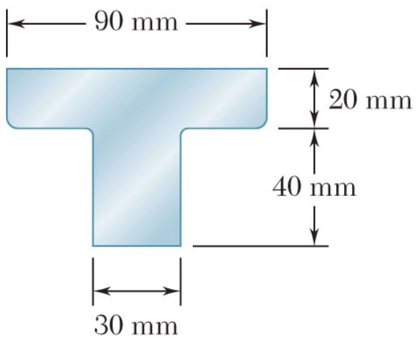
$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvature}$$

Sample Problem 4.2





A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing *E* = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + A d^2\right)$$

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

Sample Problem 4.2

SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

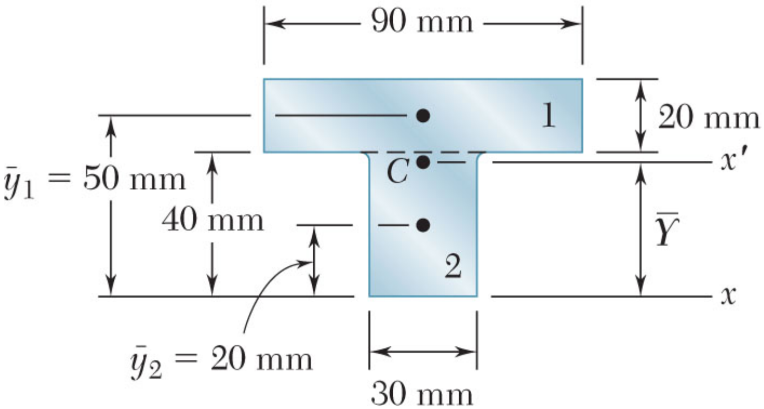


Fig. 1 Composite areas for calculating centroid.

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20 \times 90 = 1800$	50	90×10^3
2	$40 \times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114 \times 10^3$

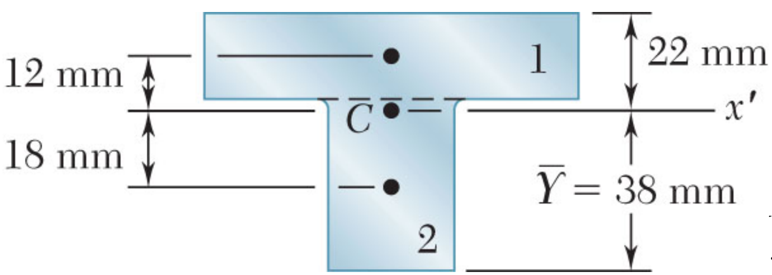


Fig. 2 Composite sections for calculating moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114 \times 10^3}{3000} = 38 \text{ mm}$$

$$I_{x'} = \sum\left(\bar{I} + Ad^2\right) = \sum\left(\tfrac{1}{12}bh^3 + Ad^2\right)$$

$$= \left(\tfrac{1}{12}90 \times 20^3 + 1800 \times 12^2\right) + \left(\tfrac{1}{12}30 \times 40^3 + 1200 \times 18^2\right)$$

$$I = 868 \times 10^3 \text{ mm}^4 = 868 \times 10^{-9} \text{ m}^4$$

Sample Problem 4.2

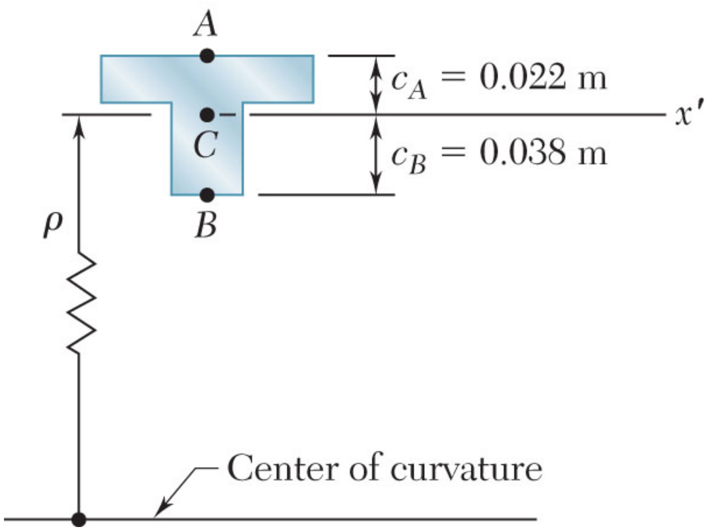


Fig. 3 Deformed radius of curvature is measured to the centroid of the cross sections.

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{M c_A}{I} = \frac{3\,\mathrm{kN\cdot m} \times 0.022\,\mathrm{m}}{868 \times 10^{-9}\,\mathrm{m}^4} \qquad \boxed{\sigma_A = +76.0\ \mathrm{MPa}}$$

$$\sigma_B = -\frac{M c_B}{I} = -\frac{3\,\mathrm{kN\cdot m} \times 0.038\,\mathrm{m}}{868 \times 10^{-9}\,\mathrm{m}^4} \qquad \boxed{\sigma_B = -131.3\ \mathrm{MPa}}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$= \frac{3\,\mathrm{kN\cdot m}}{(165\,\mathrm{GPa})\left(868 \times 10^{-9}\,\mathrm{m}^4\right)}$$

$$\boxed{\begin{aligned} \frac{1}{\rho} &= 20.95 \times 10^{-3}\,\mathrm{m}^{-1} \\ \rho &= 47.7\ \mathrm{m} \end{aligned}}$$

Eccentric Axial Loading in a Plane of Symmetry

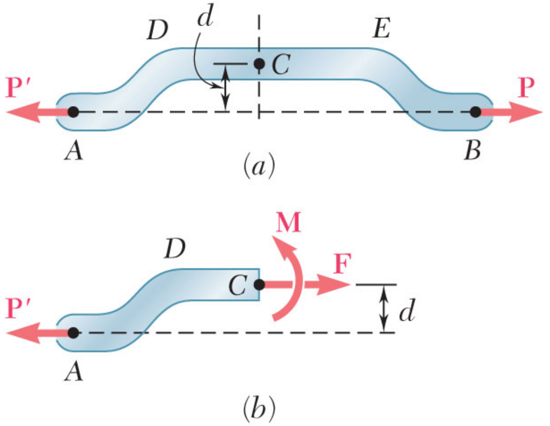


Fig. 4.39 (a) Member with eccentric loading. (b) Free-body diagram of a member with internal loads at section C.

- Eccentric loading

$$F = P$$
$$M = Pd$$

- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due to a pure bending moment

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$
$$= \frac{P}{A} - \frac{My}{I}$$

- Result are valid if stresses do not exceed the proportional limit, deformations have negligible effect on geometry, and stresses are not evaluated near points of load application.

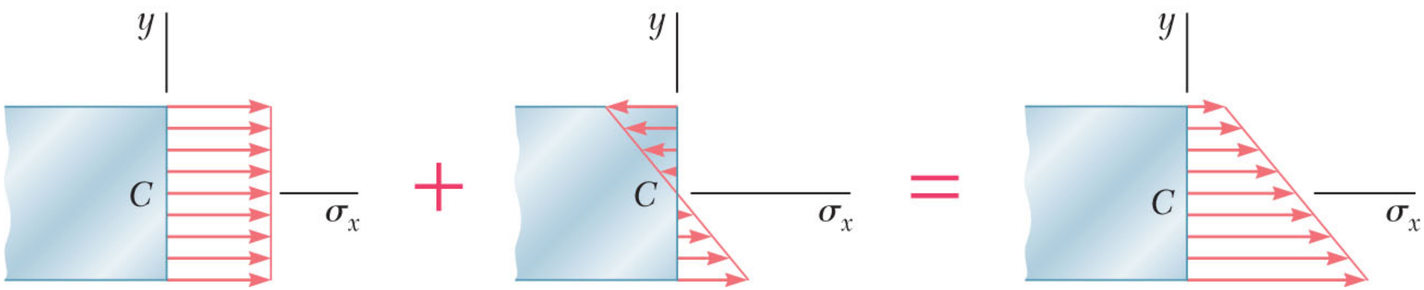


Fig. 4.41 Stress distribution for eccentric loading is obtained by superposing the axial and pure bending distributions.

Concept Application 4.7

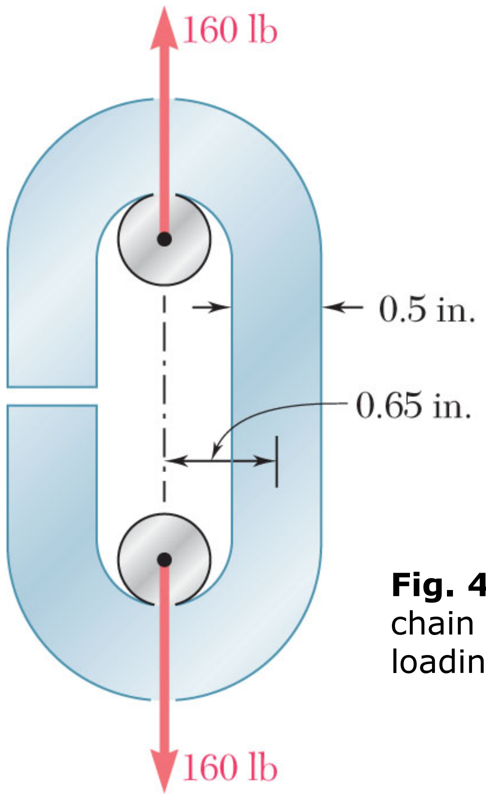


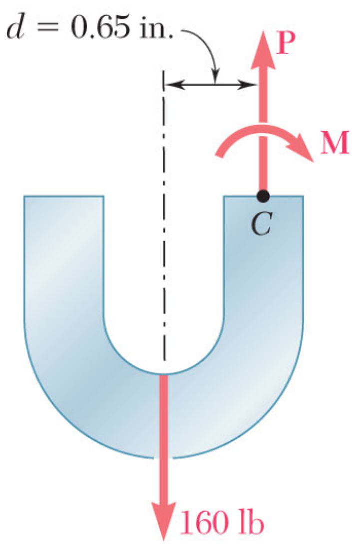
Fig. 4.43 Open chain link under loading.

An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 160 lb load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

SOLUTION:

- Find the equivalent centric load and bending moment
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Find the neutral axis by determining the location where the normal stress is zero.

Concept Application 4.7



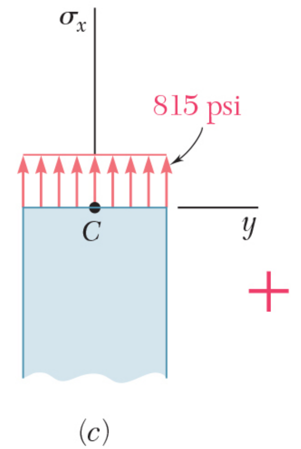
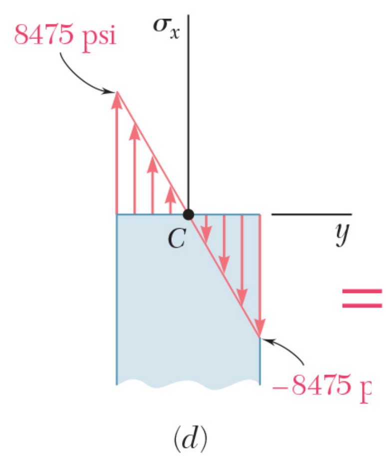


Fig. 4.43 Free-body diagram for section at C to find axial force and moment. Stress at section C is superposed axial and bending stresses.



- Equivalent centric load and bending moment

$$P = 160\,\text{lb}$$

$$M = Pd = (160\,\text{lb})(0.65\,\text{in})$$
$$= 104\,\text{lb}\cdot\text{in}$$

- Normal stress due to a centric load

$$A = \pi c^2 = \pi(0.25\,\text{in})^2$$
$$= 0.1963\,\text{in}^2$$

$$\sigma_0 = \frac{P}{A} = \frac{160\,\text{lb}}{0.1963\,\text{in}^2}$$
$$= 815\,\text{psi}$$

- Normal stress due to bending moment

$$I = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi(0.25)^4$$
$$= 3.068\times10^{-3}\,\text{in}^4$$

$$\sigma_m = \frac{Mc}{I} = \frac{(104\,\text{lb}\cdot\text{in})(0.25\,\text{in})}{3.068\times10^{-3}\,\text{in}^4}$$
$$= 8475\,\text{psi}$$

Concept Application 4.7

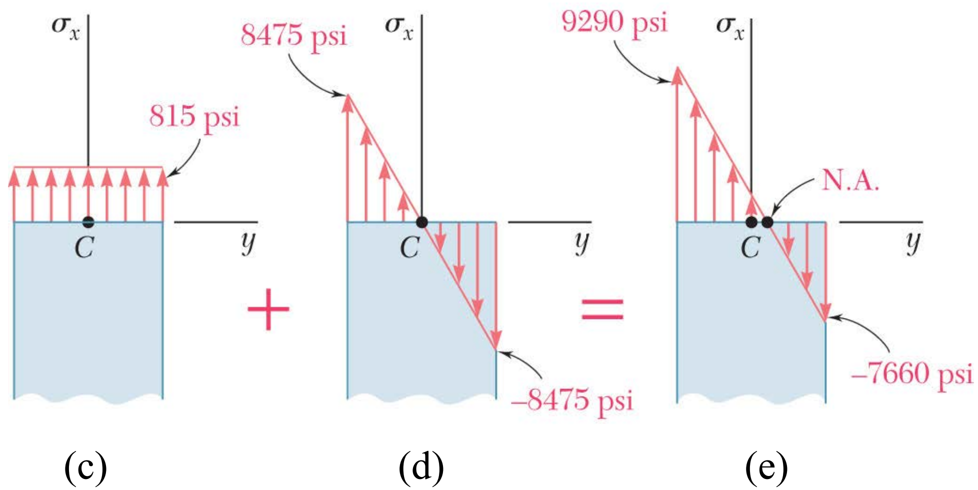


Fig. 4.43 (c) Axial stress at section C. (d) Bending stress at C. (e) Superposition of stresses.

- Maximum tensile and compressive stresses

$$\sigma_t = \sigma_0 + \sigma_m = 815 + 8475$$

$$\boxed{\sigma_t = 9260\,\text{psi}}$$

$$\sigma_c = \sigma_0 - \sigma_m = 815 - 8475$$

$$\boxed{\sigma_c = -7660\,\text{psi}}$$

- Neutral axis location

$$0 = \frac{P}{A} - \frac{My_0}{I}$$

$$y_0 = \frac{P}{A}\frac{I}{M} = (815\,\text{psi})\frac{3.068\times10^{-3}\,\text{in}^4}{105\,\text{lb}\cdot\text{in}}$$

$$\boxed{y_0 = 0.0240\,\text{in}}$$

Sample Problem 4.8

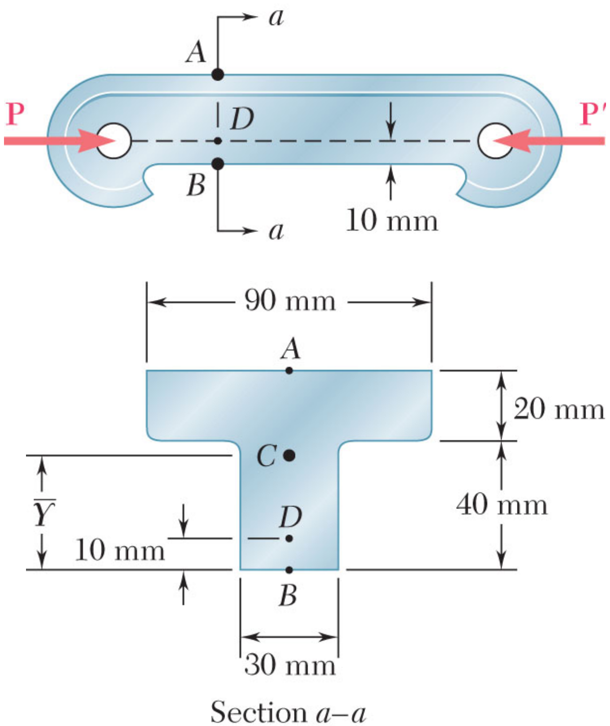


Fig. 1 Section geometry to find centroid location.

From Sample Problem 4.2,

$$A = 3\times10^{-3}\ \text{m}^2$$
$$\overline{Y} = 0.038\,\text{m}$$
$$I = 868\times10^{-9}\ \text{m}^4$$

The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force P which can be applied to the link.

SOLUTION:

- Determine equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

Sample Problem 4.8

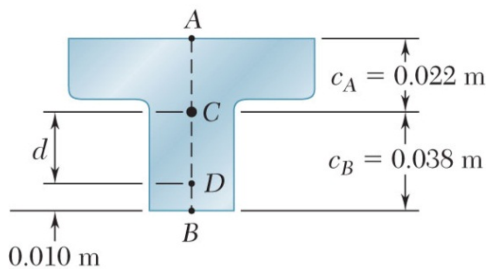
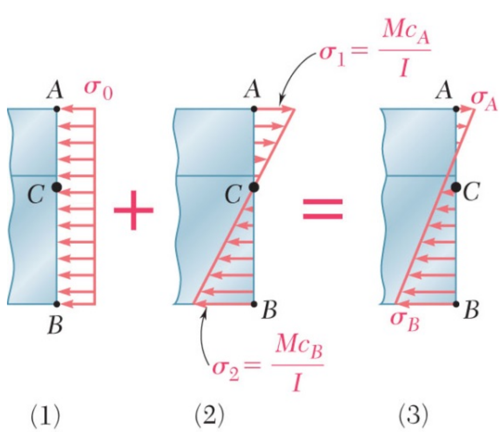


Fig. 2 Section dimensions for finding location of point D.



Figs. 4 Stress distribution at section C is superposition of axial and bending distributions acting at centroid.

- Determine equivalent centric and bending loads.

$$d = 0.038 - 0.010 = 0.028\,\mathrm{m}$$

$$P = \text{centric load}$$

$$M = Pd = 0.028\,P = \text{bending moment}$$

- Superpose stresses due to centric and bending loads

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} + \frac{(0.028\,P)(0.022)}{868\times10^{-9}} = +377\,P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} - \frac{(0.028\,P)(0.022)}{868\times10^{-9}} = -1559\,P$$

- Evaluate critical loads for allowable stresses.

$$\sigma_A = +377\,P = 30\,\mathrm{MPa} \qquad P = 79.6\,\mathrm{kN}$$

$$\sigma_B = -1559\,P = -120\,\mathrm{MPa} \qquad P = 77.0\,\mathrm{kN}$$

- The largest allowable load $\boxed{P = 77.0\ \mathrm{kN}}$