Assignment 11 Sol'ns

§7.1

(9) Let $f \in \mathcal{R}[a,b]$ and let $(\dot{\mathcal{P}}_n)$ be a sequence of tagged partitions s.t. $\|\dot{\mathcal{P}}_n\| \to 0$. Let $\varepsilon > 0$ be given. $\because f \in \mathcal{R}[a,b]$, $\exists\, \delta > 0$ s.t. $\forall$ tagged partitions $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$, $|S(f,\dot{\mathcal{P}}) - \int_a^b f| < \varepsilon$. $\because \|\dot{\mathcal{P}}_n\| \to 0$, $\exists\, K \in \mathbb{N}$ s.t. $\forall n \geq K$ $\|\dot{\mathcal{P}}_n\| < \delta$. $\therefore$ $\forall n \geq K$, $|S(f,\dot{\mathcal{P}}_n) - \int_a^b f| < \varepsilon$. $\therefore$ $\int_a^b f = \lim_n S(f,\dot{\mathcal{P}}_n)$

(11) Were $f \in \mathcal{R}[a,b]$, since $\|\dot{\mathcal{P}}_n\| \to 0$ and $\|\dot{\mathcal{Q}}_n\| \to 0$, by Q9, we would have $\lim_n S(f,\dot{\mathcal{P}}_n) = \lim_n S(f,\dot{\mathcal{Q}}_n) = \int_a^b f$. But we are told $\lim_n S(f,\dot{\mathcal{P}}_n) \neq \lim_n S(f,\dot{\mathcal{Q}}_n)$, so this contradiction shows $f \notin \mathcal{R}[a,b]$.

(12) Let $\mathcal{P}_n$ & $\mathcal{Q}_n$ both be the partition of $[a,b]$ into n equal subintervals, but for $\dot{\mathcal{P}}_n$ choose rational tags, and $\dot{\mathcal{Q}}_n$ choose irrational ones. (By the density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$, each subinterval contains both rationals and irrationals.) $\|\dot{\mathcal{P}}_n\| = \|\dot{\mathcal{Q}}_n\| = \frac{b-a}{n} \longrightarrow 0$, but $S(f,\dot{\mathcal{P}}_n) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$ $= \sum_{i=1}^n 1\,(x_i - x_{i-1}) = b-a$, while $S(f,\dot{\mathcal{Q}}_n) = \sum_{i=1}^n f(u_i)(x_i - x_{i-1})$ $= 0$ $\because f(u_i) = 0\ \forall i$. ($t_i$ are the tags for $\dot{\mathcal{P}}_n$, u_i the tags for $\dot{\mathcal{Q}}_n$)