Assignment 9 Solns

5.4.2 Let $f(x) = 1/x^2$, $A = [1, \infty)$, $B = (0, \infty)$. $\forall x, y \in A$,

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = |y-x| \cdot \frac{|y+x|}{x^2y^2} \le |x-y|\left(\frac{|y|}{x^2y^2} + \frac{|x|}{x^2y^2}\right) = |x-y|\left(\frac{1}{x^2|y|} + \frac{1}{|x|y^2}\right)$$

$\le 2|x-y|$. $\therefore$ for $\varepsilon > 0$ let $\delta = \varepsilon/2$. Then $\forall x, y \in A$, $|x-y| < \delta \Rightarrow$

$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \le 2|x-y| < 2\delta = \varepsilon$. $\therefore$ f is u.c. on A.

To show f is not u.c. on B, let $\varepsilon_0 = 1$, and let $x_n = \frac{1}{\sqrt{n+1}}$, $y_n = \frac{1}{\sqrt{n}}$. $\lim(x_n - y_n) = \lim(x_n) - \lim(y_n) = 0 - 0 = 0$ ($\because$ both converge).

However, $\left|\frac{1}{x_n^2} - \frac{1}{y_n^2}\right| = |n+1-n| = 1 \ge \varepsilon_0$. $\therefore$ f is not u.c. on B.

5.4.5 Let f, g be u.c on A. Let $\varepsilon > 0$ be given. $\exists \delta_1 > 0, \delta_2 > 0$, s.t. $\forall x, y \in A$, if $|x-y| < \delta_1$ then $|f(x) - f(y)| < \varepsilon/2$ and if $|x-y| < \delta_2$ then $|g(x) - g(y)| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\forall x, y \in A$, if $|x-y| < \delta$, $|f(x)+g(x) - (f(y)+g(y))|$

$$= |f(x) - f(y) + g(x) - g(y)| \underset{\Delta\text{ineq}}{\le} |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ $f+g$ is u.c. on A.