Assignment 6 Solns

17(a) Let $c \in \mathbb{R}$. By the density of the irrationals in $\mathbb{R}$, for any $\delta > 0$ $\exists$ irrational α s.t. $c < \alpha < c + \delta$. Then $\alpha \in V_\delta(c) \setminus \{c\}$. $\therefore$ any $c \in \mathbb{R}$ is a cluster point of the set of irrational numbers.

4.1.7 Let $\varepsilon > 0$ be given. $|x^3 - c^3| = |x-c|\,|x^2 + cx + c^2|$ (difference of cubes)
If we insist that $\delta \le 1$, then $|x-c| < \delta \Rightarrow |x| = |x-c+c|$
$\le |x-c| + |c| < \delta + |c| \le 1 + |c|$. Hence, if $\delta \le 1$, $|x^2 + cx + c^2|$
$\le |x|^2 + |c|\,|x| + |c|^2 \le (1+|c|)^2 + |c|(1+|c|) + |c|^2 = 1 + 3|c| + 3|c|^2$.
$\therefore$ Let $\delta = \min\left\{1, \dfrac{\varepsilon}{1 + 3|c| + 3|c|^2}\right\}$. Then for any x s.t. $0 < |x-c| < \delta$,
$|x^3 - c^3| \le |x-c|\,(1 + 3|c| + 3|c|^2) < \varepsilon$.

12(d) Let $x_n = \dfrac{1}{\sqrt{\pi/2 + n\pi}}$ for all n. Since $0 < x_n < \frac{1}{\sqrt{n}}$ and $\lim\left(\frac{1}{\sqrt{n}}\right) = \sqrt{\lim\left(\frac{1}{n}\right)} = 0$,
$x_n \to 0$ by the Squeeze theorem. Moreover, $x_n \neq 0$ $\forall n$. But
$\left(\sin\left(\frac{1}{x_n^2}\right)\right) = \left(\sin\left(\pi/2 + n\pi\right)\right) = \left((-1)^n\right)$ diverges. $\therefore$, by ~~the~~ a
divergence criterion, the limit diverges.