MATH 1009 TEST 3 - SOLUTIONS October 2014

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1. [**5 Marks**] Find the derivative of the function $f(x) = \left(\dfrac{2x}{x+1}\right)^5$.

Solution:

$$f'(x) = 5\left(\frac{2x}{x+1}\right)^4 \cdot \left(\frac{2x}{x+1}\right)' = 5\left(\frac{2x}{x+1}\right)^4 \cdot \left(\frac{2(x+1)-2x(1)}{(x+1)^2}\right) =$$

$$= 5\left(\frac{2x}{x+1}\right)^4 \cdot \frac{2}{(x+1)^2} \left(= \frac{10 \cdot 2^4 \cdot x^4}{(x+1)^6} = \frac{160x^4}{(x+1)^6}\right)$$

2. [**6 Marks**] Use implicit differentiation to find $\dfrac{dy}{dx}$ if $x^2 + 4xy + y^3 = 7$.

Solution:

Differentiate both sides with respect to x:

$$2x + 4 \cdot 1 \cdot y + 4x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

Solving for $\dfrac{dy}{dx}$ yields

$$\frac{dy}{dx} = -\frac{2x+4y}{4x+3y^2}.$$

3. [**12 Marks**] The quantity demanded each week x is related to the unit price p by the demand equation

$$x = f(p) = \sqrt{400-5p} \qquad (0 < p < 80).$$

The elasticity of demand is given by the formula $E(p) = -\dfrac{p\, f'(p)}{f(p)}$.

[9] **(a)** Is the demand elastic or inelastic at $p = 40$? at $p = 60$?
[2] **(b)** For what price p is the demand unitary?
[1] **(c)** If the unit price is increased slightly from 40, will the revenue increase or decrease?

Solution:
(a) We compute

$$f'(p) = \left(\sqrt{400-5p}\right)' = \frac{1}{2} \cdot \left(\sqrt{400-5p}\right)^{-\frac{1}{2}} \cdot (-5) = -\frac{5}{2\sqrt{400-5p}}.$$

Thus, $E(p) = -\dfrac{p \cdot f'(p)}{f(p)} = -\dfrac{p \cdot \dfrac{-5}{\sqrt{400-5p}}}{2\sqrt{400-5p}} = \dfrac{5p}{2(400-5p)}.$

$$E(40) = \frac{5 \cdot 40}{2(400 - 5 \cdot 40)} = \frac{1}{2} < 1, \text{ so the demand is inelastic when } p = 40.$$

$$E(60) = \frac{5 \cdot 60}{2(400 - 5 \cdot 60)} = \frac{3}{2} > 1, \text{ so the demand is elastic when } p = 60.$$

(b) $E(p) = \dfrac{5p}{2(400 - 5p)} = 1$ when $5p = 2(400 - 5p)$, $5p + 10p = 800$, so for $p = 800/15 \approx 53.3$ the demand is unitary.

(c) Since the demand is inelastic when $p = 40$, the revenue will increase if the unit price is increased slightly from 40.

4. [**11 Marks**] State the domain of each function given below. Find the intervals on which the function is increasing and those where it is decreasing:

[7] (a) $f(x) = x^3 - 12x + 1$. [4] (b) $g(x) = 7^{x^2}$.

Solution:

(a) $f(x)$ is a polynomial function, so its domain is the set of all real numbers.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2).$$

$f(x)$ is increasing when $f'(x) > 0$, that is, for $x < -2$ or for $x > 2$.
$f(x)$ is decreasing when $f'(x) < 0$, that is, for $-2 < x < 2$.

(b) $g(x)$ is an exponential function, defined on the set of all real numbers for which x^2 is defined, that is, $x \in \mathbb{R}$.

$$g'(x) = 7^{x^2} \cdot \ln 7 \cdot (2x).$$

Since $7^{x^2} > 0$ for all $x \in \mathbb{R}$ and $\ln 7 > 0$, the derivative $g'(x) > 0$ when $x > 0$, so the function $g(x)$ is increasing on the interval $(0, \infty)$.
The derivative $g'(x) < 0$ when $x < 0$, so the function $g(x)$ is decreasing on the interval $(-\infty, 0)$.

5. [**6 Marks**] Find the second derivative of the given functions:

[4] (a) $f(x) = \log_3(2x + 1)$, [2] (b) $g(x) = \dfrac{1}{9}e^{3x-5}$.

Solution:

(a) $f'(x) = \dfrac{1}{(2x+1)\ln 3} \cdot (2x+1)' = \dfrac{2}{(2x+1)\ln 3} = \dfrac{2}{\ln 3} \cdot (2x+1)^{-1}$.

$$f''(x) = \frac{2}{\ln 3} \cdot (-1)(2x+1)^{-2} \cdot (2x+1)' = -\frac{4}{\ln 3}(2x+1)^{-2} = -\frac{4}{\ln 3\,(2x+1)^2}.$$

(b) $g'(x) = \dfrac{1}{9}e^{3x-5} \cdot (3x-5)' = \dfrac{1}{3}e^{3x-5}$.

$$g''(x) = \frac{1}{3}e^{3x-5} \cdot (3x-5)' = e^{3x-5}.$$