Review Math 208 - Sections I & II, continued

Farhad Babaee

Concordia University

Statistically, 70% of the students in Math 208 do not get the required B- score to continue in JMSB. So, what should you do, to beat these numbers?

1. Be patient with maths, it always takes time to understand a new concept in mathematics. Learning mathematics is not about being fast, but about how deeply you understand the concepts, and once you do, it becomes very simple. Take your time with the examples, and make sure you completely understand their statements. Ask yourself what information the statement is providing for you.
2. If you want good marks in this course, do all exercises mentioned in the course outline.
3. Make sure you understand the problems in your own way, rephrase the problems, ask yourself questions and try to answer them on your own.

4. Read problem solving strategies written below carefully.
5. Remember that the best math tutor for you is yourself, however, you can use the office hours to get help with your difficulties, or use the maths tutors[1] in

   Math Help Centre: It has been organized to help students in solving problems. The location is LB 912 and the schedule is posted ▸ here

   There are also a few **Free Tutoring Groups** for Math 200-209 courses, take a look at:

   ▸ Learning-support
6. Finally, know that there's more to maths than exams, grades and methods.

[1] As mentioned in the course outline.

How to solve your financial problems?

1. Draw a timeline for payments, deposits, withdrawals, loans, etc.
2. See if it is a single-payment or multiple-payment problem.
3. If it's paid only once and there's interest, see if it is simple, compounded periodically or continuously.
4. If there are several payments, are the amounts of the payments equal?
5. If the payments are equal, is this a future value (such as sinking funds,saving accounts etc.) or present value problem (such as deposit-withdrawal or loan-repay, such as amortization problems)?
6. If the payments are different, the only way to solve the problem is to draw a timeline and try to see how much payment is added each time (just like the important Examples 6 and 7 at Page 607.).

1. Remember that the price-demand functions is linear (at least in this course).
$$p(x) = m - nx$$
where x is the number of items that can be sold at p dollars per item.
2. If you have the price function $p(x)$, where x is the number of items which can be sold at this price, the revenue function is given by
$$R(x) = \text{number of items sold} \times \text{price} = xp(x)$$
3. The cost function, $C(x)$, (if it is linear) is given by
$$C(x) = (\text{fixed costs}) + (\text{cost per item}) \times (\text{number of items}) = a + bx.$$
4. Profit is given by $P(x) = R(x) - C(x)$

5. The values for x where $P(x) = 0$ is the number of items where cost and revenue is equal, these values are called break-even values. It's basically the intersection of $y = R(x)$ and $y = C(x)$.
6. For any given cost function $C(x)$, the average cost function $\bar{C}(x)$ is just
$$\frac{C(x)}{x},$$
simple as that!

Solving problems of the arithmetic and geometric sequences

1. Ask yourself whether the question is about arithmetic or geometric sequence, writing down the first terms of the sequence is usually helpful.
2. Write the nth term as $a_n = a_1 + (n-1)d$ for the arithmetic and $a_n = a_1 r^{n-1}$ for the geometric sequence.
3. Note that an arithmetic sequence might be given by $a_n - a_{n-1} = d$ and a geometric sequence, by $\frac{a_n}{a_{n-1}} = r$.
4. Recall that the formula for the sum of the n first terms of an arithmetic sequence is

$$S_n = \frac{n \times (\text{first term} + n\text{-th term})}{2} = \frac{n(2a_1 + (n-1)d)}{2}$$

5. Recall that the formula for the sum of the n first terms of a geometric sequence is

$$S_n = a_1 \frac{1 - r^n}{1 - r}, \quad r \neq 0.$$

6. When you are asked to calculate the sum of an infinite geometric series, and you realize that $r \geq 1$ or $r \leq -1$, say that the sum is not finite, but when $-1 < r < 1$, then in the formula in item 4, r^n becomes zero, to obtain:

$$S_\infty = a_1 \frac{1}{1 - r}.$$

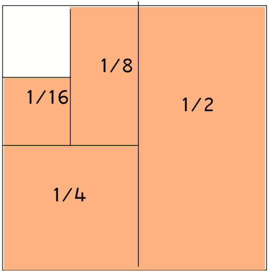

Figure: The sum $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ tends to one, since the rectangles are filling up the whole square, check this using the formula.

Compounded interest

Example 1

If \$1000 is invested in an account that earns 9.75% compounded annually for 6 years, find the interest earned during each year and the amount in the account at the end of each year. Organize your results in a table.

Period	Interest	Amount
0	\$0	\$1,000.00
1	\$97.50	\$1,097.50
2	\$107.50	\$1,204.51
3	\$117.44	\$1,321.95

Example 2

How long will it take \$4,000 to grow to \$9,000 if it is invested at 7% compounded monthly? $(11\frac{2}{3} yr)$

Example 3

You have saved \$7000 toward the purchase of a car costing \$9000. How long will the \$7000 have to be invested at 9% compounded monthly to grow to \$9000?

What bank account do you prefer?

1. One with annual rate $r = 10\%$, and the future value is calculated with simple interest.
2. One which has annual rate $r = 10\%$ and the future value is compounded semiannually.
3. One which has annual rate $r = 10\%$ and the future value is compounded daily.
4. One which has annual rate $r = 10\%$ and the future value is compounded secondly.
5. One which has annual rate $r = 10\%$ and the future value is compounded continuously.

Continuously compounded interest

You may rewrite,

$$P(1+\frac{r}{m})^n$$

as

$$P(1+\frac{r}{m})^{mt}$$

where $t=\frac{n}{m}$, is the time in years. Since $(1+\frac{r}{m})^{mt}$ is like e^{rt}, when m is very large. We have,

$$P(1+\frac{r}{m})^{mt} \text{ is close to } Pe^{rt}.$$

Theorem 4

Future value of a continuously compounded interest is given by

$$A=Pe^{rt},$$

where r is the nominal annual interest, and t is the time in years.

Example 5

How long will it take \$42,000 to grow to \$60,276 if it is invested at 4.25% compounded continuously?

Example 6

A promissory note will pay $\$50,000$ at maturity 6 years from now. If you pay $\$28,000$ for the note now, what rate compounded continuously would you earn?

Annual Percentage Yield

Definition 7

The interest that you receive after one year by depositing \$1 in a given account is called Annual Percentage Yield (APY) of that investment. Remember to write this number in percentage form.

Example 8

What is the APY of an account that pays 12% compounded semi-annually.

Solution. If you deposit \$1 in this account, after one year it becomes

$$1(1+\frac{0.12}{2})^2 = 1.123600000.$$

Therefore your one dollar becomes \$ 1.1236, the extra $0.1236 = 12.36\%$ is the percentage which this account yields annually.

You may also think of APY of an account as the simple interest rate which yields the same amount of interest as that account.

Example 9

What's the APY for an account which pays 5% compounded continuously?

Solution. Your one dollar, after one year, in this account becomes $e^{0.05}$, therefore, the extra money $e^{0.05} - 1 = 0.051271096 = 5.1271096\%$ is the APY. This means that if you deposit your one dollar in another bank which pays the annual interest of 5.1271096%

Sometimes, it is better to write $i = \frac{r}{m}$, and then i becomes the interest rate for each compounding period. For example, if they say annual rate is $r = 10\% = 0.10$, then the weekly interest rate is $i = \frac{0.10}{54}$. In some questions (the loan shark question), we are directly given the interest rate for one compounding period. For instance, in that example the interest rate per week was $i = 1\% = 0.01$ and we calculated

$$500(1 + 0.01)^{26}$$

as the future value after 26 weeks. Note that, the annual interest rate $r = (54)(0.01)$.

So, for calculating the compound interest rate instead of

$$A = P(1 + \frac{r}{m})^n.$$

we can also simply use the formula:

$$A = P(1 + i)^n.$$

where i is the interest per each compounding periods. This formula is usually written on the exam sheet, but you should know the meaning of each parameter in the formula.

Future Value of an Annuity

Definition 10

An **annuity** is any sequence of equal periodic payments, which are usually paid at the end of each period (this is our assumption throughout this course).

Definition 11

The amount or **future value** of an annuity is the sum of all payments plus all interest earned.

Example 12

If you deposit \$100 every year (at the end of each year), for 5 years in an account, how much would be the future value of your deposits if the interest is calculated every year at annual rate of $r = 10\%$?

Solution. The interest is calculated every year at annual rate of $r = 10\%$?

1. At the end of the first year we deposit \$ 100, after four years (five years time from now) the value is $100(1+0.1)^4$. (Why?)
2. At the end of the second year we deposit \$100, after 3 years (five years time from now) the value of that \$100 is $100(1+0.1)^3$.
3. At the end of the third year we deposit \$100, after 2 years (five years time from now) the value of that \$100 is $100(1+0.1)^2$.
4. At the end of the forth year we deposit \$100, after 1 year (five years time from now) the value of that \$100 is $100(1+0.1)^1$.
5. At the end of the fifth year we deposit \$100, after 0 years (five years time from now) the value of that \$100 is $100(1+0.1)^0 = 100$. Therefore the future value of our annuity is

$$100(1.1)^4 + 100(1.1)^3 + 100(1.1)^2 + 100(1.1)^1 + 100(1.1)^0.$$

How to calculate

$$100(1.1)^4 + 100(1.1)^3 + 100(1.1)^2 + 100(1.1)^1 + 100(1.1)^0?$$

This is a geometric sequence with five terms, with $a = 100$ and $R = (1.1)$. We know that

$$a + aR + aR^2 + aR^3 + aR^4 = a(\frac{R^5 - 1}{R - 1}).$$

$$a = 100(1.1)^0 = 100, aR = 100(1.1)^1, aR^2 = 100(1.1)^2$$

$$aR^3 = 100(1.1)^3, aR^4 = 100(1 + 0.1)^4.$$

We have

$$a(\frac{R^5 - 1}{R - 1}) = (100)\frac{(1.1^5 - 1)}{0.1} = 610.51.$$

Future Value of an (Ordinary) Annuity

Theorem 13

If the payments are made at the end of each period then

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

FV = *future value*
PMT = *periodic payment*
i = *rate per period*
n = *number of payments (periods)*
Warning: This formula works only when the payments are made at the end of each compounding period! [a]

[a] This formula will be provided on the exam.

In the previous example $PMT = 100, i = 0.1, n = 5$. Therefore $FV = (100)\frac{(1.1^5-1)}{0.1} = 610.51$ How much is the interest earned? $610.51 - 500 = 110.51$

Example 14 (Sinking funds)

How much should parents deposit each year, for 18 years in an account that pays 6%, compounded annually, so that they accumulate $100,000 for their child?

Solution.

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

$i = r = 0.06$, $n = 18$, $FV = 100000$, gives
$PMT = 3235.654055$.
How much is the total interest gained?
100,000 - 3235.65*18= 41758.30.

Example 15

A company estimates that it will need $\$100,000$ in 8 years to replace a computer. If it establishes a sinking fund by making fixed monthly payments into an account paying 7.5% compounded monthly, how much should each payment be?

Note is the interest is paid compounded monthly and we pay every month. [Because our formula works only here.]

Example 16 (Not covered in the book, and you may not use the above formula)

If we deposit 1000 every year for 4 years, in an account that pays 10% compounded monthly how much is the future value after 4 years?

Example 17

Beginning in January, a person plans to deposit \$100 at the end of each month into an account earning 6% compounded monthly. Each year taxes must be paid on the interest earned during that year. Find the interest earned during each year for the first 3 years (taxes are paid from a separate account).

Solution. How much is deposited after the first year? 1200
How much is the future value of the annuity after one year?
$100\frac{(1+\frac{0.06}{12})^{12}-1}{\frac{0.06}{12}} = 1233.556240$
How much is the interest earned? 33.556240
\$1233.56 after one year (in the end of the second year) will be
$1233.56(1+\frac{0.06}{12})^{12} = 1309.639290$

The interest earned for this amount, after a year, is therefore 1309.639290-1233.56 =76.079290. The deposits of $100 in this year yield $ 33.556240, therefore the whole interest which is earned in the second year (before reducing the tax) is:

$$76.079290 + 33.556240 = 109.635530$$

How do we calculate the interest earned during the third year?

Present value

Assume that the safest bank in Canada pays 5% annual interest. What would you prefer (if you don't urgently need money)?

1. Being awarded \$1000, or
2. Being awarded \$1100 next year?

Present value of \$1000 at hand is \$1000 itself.

Present value of that \$1100 which you will have next year is $\frac{1100}{1+0.05} = 1047.61$. Meaning that \$1047.61 today is worth \$1100 after one year. *i.e.* the Present Value of \$1100 of one year ahead is \$1047.61.

How much is the present value of \$1100 if it's earned after 3 years?

$$1100 = P(1+0.05)^3 \implies P = 1100(1.05^{-3}).$$

Example 18 (Present value of an Annuity, P.153)

If we have \$5,417.19 in a bank paying 6% compounded annually we are able to withdraw \$1000 every six months for the next 3 years. Why?

The rate per period is $i = 0.03$.

$\frac{\$1000}{1.03} = 1000(1.03^{-1}) = 970.87$ dollars now, will be worth \$1000 after six month. Therefore, to be able to withdraw \$1000 after six months, we should at least deposit 970.87.

$\frac{\$1000}{1.03^2} = 1000(1.03^{-2}) = 942.60$ dollars now, will be worth \$1000 after two periods of six months.

Therefore if we deposit $970.87 + 942.60$ dollars now, we will be able to withdraw \$1000 after six months and \$1000 after one year.

$\frac{1000}{1.03^2} = 1000(1.03^{-2})$ dollars now, will be worth \$1000 after two periods of six months.

$\frac{1000}{1.03^3} = 1000(1.03^{-3})$ will be \$1000 after 18 months.

$\frac{\$1000}{1.03^4} = 1000(1.03^{-4})$

$\frac{\$1000}{1.03^5} = 1000(1.03^{-5})$
$\frac{\$1000}{1.03^6} = 1000(1.03^{-6})$

We have

$$\sum_{j=1}^{6} 1000(1.03)^j = 1000(1.03^{-1}) + \cdots + 1000(1.03^{-6}) = 5,417.19.$$

Theorem 19 (Present value of an Ordinary Annuity)

$$PV = PMT\frac{1-(1+i)^{-n}}{i}$$

PV = present value of all payments
PMT = periodic payment
i = rate per period
n = number of periods
Note: Payments are made at the end of each period and that the formula is provided on the exam.

Example 20 (Retirement Planning, P. 155)

Lincoln Benefit Life offered an ordinary annuity that earned 6.5% compounded annually. A person plans to make equal annual deposits into this account for 25 years and then make 20 equal annual withdrawals of $$25000, reducing the balance in the account to zero. How much must be deposited annually to accumulate sufficient funds to provide for these payments? How much total interest is earned during this entire 45-year process?

How much will be the present value that allows 20 annual withdrawals of $25000?
How much do we have to deposit each year to accumulate that amount in 25 years?

$$PV = PMT\frac{1-(1+i)^{-n}}{i} = 275,462.68.$$

$$FV = PMT\frac{(1+i)^{n}-1}{i} \implies PMT = 4,677.76.$$

Amortization

Think!

In Example 15, we saw that if we deposit (\$5,417.19)

$$10001.03^{-1}+10001.03^{-2}+10001.03^{-3}+\cdots+10001.03^{-6}=5,417.19.$$

in our account, we'll be able to withdraw \$1000 every six months for three years. Therefore, if we borrow \$5,417.19 from our bank, we are able to retire the debt (amortize the debt), if we pay 6 payments of \$1000 every six months for three years.

Amortizing (killing) a debt means that the debt is retired in a given length of time by equal periodic payments that include compound interest.

$$PV = PMT\frac{1-(1+i)^{-n}}{i} \quad \text{Amortization Formula}$$

is the same as Present Value of an Ordinary Annuity Formula.

Example 21 (Example 4, P.157)

If you borrow \$500 that you agree to repay in six equal monthly payments at 1% interest per month on the unpaid balance, how much of each monthly payment is used for interest and how much is used to reduce the unpaid balance?

How much is paid monthly? \$86.27
How much is the total amount to be paid? \$517.65
How much is the interest for the \$500 after the first month?
500*0.01= 5
How much of the first payment is, in fact, paid to reduce the unpaid balance? 86.30 - 5= 81.30
Table 1, Page 158.

Example 22 (Example 5, Page 158)

A family purchased a home 10 years ago for $80,000. The home was financed by paying 20% down and signing a 3–year mortgage at 9% on the unpaid balance. The net market value of the house (amount received aftr substracting all costs involved in selling the house) is now $120,000, and the family wishes to sell the house. How much equity does the family have in the house now after making 120 monthly payments?

Recall that

Equity = Current net market value − unpaid load balance.

Ask yourself these questions:

How much they was the loan from the bank 10 years ago? They paid 20% down[2], which means they had to borrow 80% of the whole price of the house from the bank.

[2]the initial upfront portion of the total amount due and it is usually given in cash at the time of finalizing the transaction

Therefore they borrowed

$$80/100 * 80,000 = 64,000.$$

It is a 30-year mortgage (=360 months), compounded monthly at $\frac{0.09}{12}$0.0075% per month. Therefore we have all the information to calculate the amount of payment. What formula should we use? Are all the payments equal?
Next step is to ask yourself how much the mortgage is worth now (future value), when you made the monthly payments for 10 years, and therefore, $n = 20 * 12$ months is left to be paid.