MAT 1341E Midterm 2, 2011

19-November, 2011.
Instructor: Sanghoon Baek

Family Name:______________________

First Name:________________________

Student number:____________________

Multiple choice answers → (1–3)

For the marker's use only → (4–6)

1	
2	
3	
4	
5	
6	
[Bonus] 7	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 80 minutes to complete this exam. Read each question carefully.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones, pagers or any text storage or communication device **is not permitted.**
3. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. Please record your answers in the space provided above.
4. Questions 4 – 6 require a complete solution, and are worth 6 points each, so spend your time accordingly.
5. Question 7 is a bonus question and is worth 4 points. To earn points here will be more difficult than in questions 1-6.
6. **The correct answer in questions 4–7 requires justification written legibly and logically: you must convince the marker that you know why your solution is correct. You must answer these questions in the space provided.** Use the backs of pages if necessary.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. For what value of α is the set of vectors $\{(1,1,1),(1,2,0),\ (2,3,\alpha)\}$ linearly dependent?

A. -1
B. 2
C. 0
D. 1
E. -1/2
F. -2

: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & \alpha \end{bmatrix}$. Then we have a REF of A: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \alpha - 1 \end{bmatrix}$. If $\alpha - 1 = 0$, then $\text{rank}(A) = \dim(\text{row}(A)) = 2$, which means that the given set of vectors is linearly dependent. Hence the answer is D.

2. If A is a 7×12 matrix, what is the smallest possible dimension of $\text{null}(A)$?

A. 0
B. 3
C. 5
D. 7
E. 11
F. 12

: As $\text{rank}(A) + \dim(\text{null}(A)) = 12$ and $0 \leq \text{rank}(A) \leq 7$, we have

$$5 \leq \dim(\text{null}(A)) \leq 12.$$

Hence, the answer is C.

3. If $B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, then the second row of B^{-1} is:

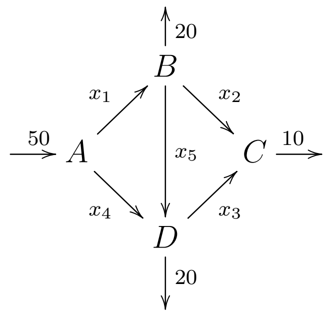
A. $[1 \;\; 0 \;\; -1]$
B. $[-1 \;\; 0 \;\; 1]$
C. $[0 \;\; 1 \;\; -1]$
D. $[2 \;\; 0 \;\; -1]$
E. $[1 \;\; -1 \;\; 0]$
F. None of the above

: We apply Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 2 | 1 & 0 & 0 \\ 1 & 3 & 1 | 0 & 1 & 0 \\ 1 & 3 & 2 | 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 3 & 0 & -2 \\ 0 & 1 & 0 & | -1 & & 0 & 1 \\ 0 & 0 & 1 & | & 0 & -1 & 1 \end{bmatrix}.$$

Hence, the answer is B.

4. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter A or leave B, C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) Write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

A: $50 = x_1 + x_4$,

B: $x_1 = 20 + x_2 + x_5$,

C: $x_2 + x_3 = 10$,

D: $x_4 + x_5 = x_3 + 20$,

where x_1, x_2, x_3, x_4, x_5 are all positive integers.

(b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 1 & 1 & 30 \\ 0 & 0 & 1 & -1 & -1 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

: Let $x_4 = s$ and $x_5 = t$. Then, we have

$$(x_1, x_2, x_3, x_4, x_5) = (-s + 50, -s - t + 30, s + t - 20, s, t).$$

(c) If $\overline{BD}$ is closed, find the maximum and minimum flows along $\overline{BC}$

: As $x_5 = t = 0$, we have $x_1 = -s + 50 \geq 0$, $x_2 = -s + 30 \geq 0$, $x_3 = s - 20 \geq 0$, and $x_4 = s \geq 0$. Hence,

$$20 \leq s \leq 30.$$

Therefore, $x_2 = -s + 30$ has the maximum 10 when $s = 20$, and has the minimum 0 when $s = 30$.

5. Let $W = \text{span}\{(1,0,1,1),(-1,1,2,0),(1,1,4,2),(0,1,3,1)\}$, and define a matrix A by

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

(a) Find a basis for W which is a **subset** of the given spanning set.

(b) Extend your basis in part (a) to a basis of $\mathbf{R}^4$.

(c) Find a basis for null(A) and hence find its dimension.

(d) Extend your basis of null(A) to a basis of $\mathbf{R}^4$.

: (a) Let $W = \{v_1, v_2, v_3, v_4\}$. Let $B = \begin{bmatrix} -v_1- \\ -v_2- \\ -v_3- \\ -v_4- \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$. Then we have a REF of B: $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence, $\{v_1, v_4\} = \{(1,0,1,1),(0,1,3,1)\}$ is a basis of W.

(b) Add $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ to $\{(1,0,1,1),(0,1,3,1)\}$. Then they form a basis of $\mathbf{R}^4$ as the rank of the matrix formed by these vectors is 4.

(c) By applying Gaussian elimination to A, we have $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Let $x_3 = s$ and $x_4 = t$. Then, we get

$$(x_1, x_2, x_3, x_4) = (-2s - t, -s - t, s, t) = s(-2,-1,1,0) + t(-1,-1,0,1).$$

Therefore, $\{(-2,-1,1,0),(-1,-1,0,1)\}$ forms a basis of null(A) and $\dim(\text{null}(A)) = 2$.

(d) Add $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$ to $\{(-2,-1,1,0),(-1,-1,0,1)\}$. Then they form a basis of $\mathbf{R}^4$ as the rank of the matrix formed by these vectors is 4.

Space for Question 5

6. Suppose A is an $n \times n$ matrix and that there is a vector $b \in \mathbf{R}^n$, for which $Ax = b$ is inconsistent.

State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example (with numbers!).
- If you say the statment is true, you must give a clear explanation.

(a) The system $Ax = 0$ has a unique solution.

False

: Let A be the zero matrix and $b = e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^n$. Then $Ax = b$ is inconsistent, but $Ax = 0$ has infinitely many solutions.

(b) The matrix A is not invertible.

True

: A is invertible if and only if $Ax = b$ is consistent for any $b \in \mathbf{R}^n$.

(c) The rows of A are linearly independent.

False

: With the same A and b as in (a) this is false.

7. [Bonus]. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose that $ad - bc = 0$. Show carefully that A is not invertible.

(Solution 1)

: Suppose that A is invertible. Then $AB = I_2$, where $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Therefore, we have $ae + bg = 1$, $af + bh = 0$, $ce + dg = 0$, $cf + dh = 1$, and $ad = bc$. As

$$0 = d(af + bh) = (ad)f + bdh = (bc)f + bdh = b(cf + dh) = b$$

and

$$0 = a(ce + dg) = c(ae + bg) = c,$$

we have $ad = 0$. This implies that $a = 0$ or $d = 0$. But this contradicts to $ae = 1$ and $dh = 1$. Hence, A is not invertible.

(Solution 2)

: First, note that A is invertible if and only if $\operatorname{rank}(A) = 2$. If $a = b = c = d = 0$, then $A = 0$. Therefore, A is not invertible. We may assume that $a \neq 0$ using elementary operations. We apply Gaussian elimination to A. Then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 0 \end{bmatrix}.$$

Hence, the rank of A is 1. This implies that A is not invertible.