Math 419/546 Assignment 1-due Fri. Jan. 16

1. Let ϕ be a non-negative Borel-measurable function on $\mathbb{R}_+$ s.t. $\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty$. If $\sup_{i\in I} E(\phi(|X_i|)) < \infty$, show that $\{X_i : i \in I\}$ is u.i.

2. p. 263 Ex. 5.5.8

3. Let $\{Y_i : i \in \mathbb{N}\}$ be independent r.v.'s with mean 0 and s.t. $|Y_i| \leq \alpha$ for all i. If $S_n = \sum_{i=1}^n Y_i$, prove that either
(a) S_n converges a.s. or (b) $P(\limsup_n S_n = +\infty \text{ and } \liminf_n S_n = -\infty) = 1$.

4. Let $\{X_n : n \in \mathbb{N}\}$ be non-negative r.v.'s and set $D = \{X_n = 0 \text{ for some } n \geq 1\}$. Assume there are $\delta_k > 0$ s.t. $P(D|\mathcal{F}_n^X) \geq \delta_k$ a.s. on $\{X_n \leq k\}$ for all $k, n \in \mathbb{N}$. Prove that w.p.1 either $X_n = 0$ for some n, or $\lim_n X_n = \infty$.
Hint. Consider $\lim_n P(D|\mathcal{F}_n^X)$.

5. Assume $\Pi_n = \{C_i^n : i \leq m_n\}$ are finite partitions of Ω s.t. Π_{n+1} refines Π_n for all $n \in \mathbb{Z}_+$ and $\mathcal{F}_n = \sigma(\Pi_n)$ satisfies $\vee_n \mathcal{F}_n = \mathcal{F}$. Let P and Q be a probabiity and a finite measure, respectively, on $(\Omega, \mathcal{F})$. Define

$$X_n(\omega) = \begin{cases} \frac{Q(C_i^n)}{P(C_i^n)}, & \text{if } \omega \in C_i^n \text{ and } P(C_i^n) > 0; \\ 0, & \text{if } \omega \in C_i^n \text{ and } P(C_i^n) = 0. \end{cases}$$

We will verify in class that X is an $\mathcal{F}_n$-supermartingale, is an $\mathcal{F}_n$- martingale iff $Q|_{\mathcal{F}_n} \ll P|_{\mathcal{F}_n}$ for all n, in which case $X_n = \frac{dQ|_{\mathcal{F}_n}}{dP|_{\mathcal{F}_n}}$, and is a u.i. $\mathcal{F}_n$-martingale if $Q \ll P$. So you may assume these results.

If $\{X_n\}$ is a u.i. $(\mathcal{F}_n)$-martingale, show conversely that $Q \ll P$ and X_n converges to $\frac{dQ}{dP}$ a.s. and in L^1.