MATH2007B - Test 4 - 14:35 -15:25, Nov 18, Tuesday

Name: Student Number:

Total: 15 marks
Closed book, Non-programmable calculators are allowed!

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1. (6 points) Multiple choices, no partial marks.

(i) Given the sequence $a_n = \frac{5^n}{9^n}$, then $\lim_{n\to\infty} a_n =$

(a) 5 (b) 17/3 (c) 5/9 (d) 0 (e) Does not exist.

Solution: (d) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{5}{9}\right)^n = 0.$

(ii) Given the sequence $\{b_n\}$: $b_1 = 2$, $b_{n+1} = \frac{1}{3}(b_n + 6)$. Then $\lim_{n\to\infty} b_n =$

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) 3 (d) 5 (e) $\frac{7}{3}$

Solution: (c). Let the limit be L. Then by the recursive relation, $L = \frac{1}{3}(L+6) \Rightarrow L = 3$.

(iii) Given the sequence $c_n = \frac{\ln n}{n}$, then $\lim_{n\to\infty} c_n =$

(a) 0 (b) 1 (c) 2 (d) e (e) Does not exist.

Solution: (a).

By L'Hospital's rule, $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = 0 \Rightarrow, \lim_{n\to\infty} c_n = 0.$

(iv) $\sum_{n=4}^{\infty} \frac{1}{n(n+1)} =$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) 1 (e) $+\infty$.

Solution: Note that

$$\frac{1}{(n)(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

$$S_n = \sum_{j=4}^{n} \frac{1}{(j)(j+1)} = \sum_{j=4}^{n} \left(\frac{1}{j} - \frac{1}{j+1}\right)$$

$$= \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + ... + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = \frac{1}{4} - \frac{1}{n+1}.$$

Therefore

$$\lim_{n\to\infty} S_n = \frac{1}{4}.$$

2. (2 points) Find all the values of k such that the series $\sum_{n=1}^{\infty} n^{2k-4}$ is convergent.

Solution: By p-series test, $-(2k-4) > 1$, $2k < 3$, $k < \frac{3}{2}$.

3. (3 points) Find the sum of the following series: $\sum_{n=1}^{\infty} \frac{1+4^n}{9^n}$.

Solution:

$$\sum_{n=1}^{\infty} \frac{1+4^n}{9^n} = \sum_{n=1}^{\infty} \frac{1}{9^n} + \sum_{n=1}^{\infty} \frac{4^n}{9^n} = \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n = \frac{\frac{1}{9}}{1-\frac{1}{9}} + \frac{\frac{4}{9}}{1-\frac{4}{9}} = \frac{37}{40} = 0.925.$$

4. (4 points) Using **Integral Test**, determine if the following series converges or diverges: $\sum_{n=3}^{\infty} \frac{2}{n(\ln n)^3}$.

Solution: We use Integral Test. Let $f(x) = \frac{2}{x(\ln x)^3}$, $x \geq 3$. Then $f(x) > 0$, and continuous.

$$f'(x) = \frac{-2[(\ln x)^3 + 3(\ln x)^2]}{x^2(\ln x)^6} < 0,$$

$f(x)$ is decreasing. By substitution with $u = \ln x$, we have

$$\begin{aligned} \int_3^{\infty} \frac{2}{x(\ln x)^3} dx &= \lim_{b\to\infty} \int_3^b \frac{2}{x(\ln x)^3} dx \\ &= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{2}{u^3} du \\ &= \left(\frac{1}{\ln 3}\right)^2. \end{aligned}$$

Hence it is convergent. By Integral Test, the series is convergent.